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TVERBERG'S THEOREM VIA NUMBER FIELDS

BY

K.S. SARKARIA

Department of Mathematics
Panjab University, Chandigarh 160014, India

ABSTRACT

We show that Tverberg's theorem follows easily from a theorem of which Bárány [1] has given a very short proof.

1. Introduction

The following fundamental fact concerning convexity was conjectured by Birch [2] in 1959, and established by Tverberg [4] in 1966. Though Tverberg's original argument was very involved, he subsequently gave a somewhat simpler proof in [5]. An alternative proof is given also in Tverberg-Vrećica [6].

TVERBERG'S THEOREM: *Any cardinality $(m+1)(q-1)+1$ subset of $\mathbb{R}^m$ can be partitioned into q disjoint subsets having a common point in their convex hulls.*

The object of this note is to give yet another proof of this result.

Our starting point will be another important fact concerning convexity, of which Bárány [1] has given a very short and elegant proof.

BÁRÁNY'S THEOREM: *Any $d+1$ subsets S_i of $\mathbb{R}^d$, whose convex hulls contain the origin, admit a* section *T (i.e. a cardinality $d+1$ set having one point from each S_i) whose convex hull also contains the origin.*

We will deduce Tverberg's theorem from Bárány's, using only some elementary facts regarding finite field extensions of the rationals.

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2. Proof of Tverberg's theorem

We first note that

Bárány's theorem remains true if, instead of $\mathbb{R}^d$, we work within the rational space $\mathbb{Q}^d$.

This follows easily from the theorem for $\mathbb{R}^d$, using the fact that if a nonzero rational vector is some real linear combination of linearly independent rational vectors, then the coefficients occuring in this linear combination, being given by Cramer's rule, are necessarily rational.

Now let $\Omega = \{v_0, v_1, \ldots, v_{(m+1)(q-1)}\}$ be a subset of the real affine m-space $\mathbb{R}A^m \subset \mathbb{R}^{m+1}$, where $\mathbb{R}A^m$ is given by $x_1 + \cdots + x_{m+1} = 1$.

In case Ω can not be partitioned into q parts having a common point in their convex hull, then the same would be true of any neighbouring cardinality $(m+1)(q-1)+1$ subset of $\mathbb{R}A^m$.

So it suffices to show the existence of such a partition only for the case when Ω is a subset of the rational affine m-space $\mathbb{Q}A^m \subset \mathbb{Q}^{m+1}$.

We choose, e.g. by using Eisenstein's criterion, some irreducible polynomial

$$a_0 + a_1X + \cdots + a_{q-1}X^{q-1} \in \mathbb{Z}[X],$$

with all a_r's nonzero integers.

Now, for each $v_i \in \Omega$, consider the cardinality q set S_i, whose elements are the vectors $a_r\omega^r v_i$, $0 \le r \le (q-1)$, where ω is a complex root of the above polynomial. The irreducibility of our polynomial shows that $(\mathbb{Q}(\omega))^{m+1}$, of which all these S_i's are subsets, has $\mathbb{Q}$-dimension $d = (m+1)(q-1)$.

Further, since $\sum_{r=0}^{q-1} a_r\omega^r v_i = 0$ for each i, the origin 0 is contained in the convex hull of each of these $d+1$ sets S_i.

Applying the above rational analogue of Bárány's Theorem, to this rational d-dimensional space, we get non-negative rationals t_i, with sum 1, such that

$$\sum_{i=0}^{(m+1)(q-1)} t_i.a_{n_i}.\omega^{n_i}.v_i = 0,$$

for suitable choices $0 \le n_i \le q-1$.

So Ω partitions into the sets

$$\Omega_r = \{v_i : n_i = r\}, \quad 0 \le r \le q-1,$$

such that

$$\sum_{r=0}^{q-1}\Big(\sum_{v_i\in\Omega_r} t_i.v_i\Big)a_r\omega^r = 0.$$

Since the a_r's of our irreducible polynomial were nonzero, it follows that the rational vectors $\sum_{v_i\in\Omega_r} t_i.v_i$ are all equal to each other. Likewise, for the same reason, the sums $\sum_{v_i\in\Omega_r} t_i$ of their $m+1$ coordinates, are all equal to each other, and so equal to a positive rational number.

Thus the vectors

$$\frac{\sum_{v_i\in\Omega_r} t_i.v_i}{\sum_{v_i\in\Omega_r} t_i}$$

are equal to each other, and provide us with a common point of all the convex hulls $conv(\Omega_r)$, $0\le r\le q-1$. ∎

3. Remarks

(1) The above proof gives a conceptual insight into the number occuring in the Tverberg-Birch theorem, and so leads further to many nice global and topological generalizations, which will be included in Chapter IV (on "Linear Embeddability") of [3].

(2) I would like to thank C.S.Yogananda for telling me about an interesting problem, involving an *integral equiangular polygon*, which was posed in the Beijing Mathematics Olympiad, 1990.

Since this problem led to the above proof, and to some other results of [3], it seems worth pointing out that for the polynomial $1+X+\cdots+X^{p-1}$, p prime, its irreducibility (the property employed in the above proof) is equivalent to the geometrical fact that *for p prime, an equiangular p-gon having rational sides is necessarily regular.*

As against this, for q composite, there can be many non-regular integral equiangular q-gons, but the interesting problem of obtaining a complete classification of such q-gons seems to be still open.

(3) However, after learning about the above proof from me, and guided by the fact that the particular values of the nonzero coefficients a_r's played no role whatsoever, *S.Onn was able to modify the above proof of Tverberg's theorem into one which avoids rationality questions altogether*:

To do this consider $\mathbb{R}^d$ as the space of $(m+1)\times(q-1)$ matrices; for each $v_i\in\Omega$ let $S_i\subset\mathbb{R}^d$ be the cardinality q set of which $q-1$ of the members are column vectors which are copies of v_i, and the last member is the matrix having $-v_i$ in every column; now apply the (real) Bárány theorem and proceed as before.

It seems that both the rational and real versions will illuminate the way towards many interesting algebraical generalizations of Tverberg's theorem.

References

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