ਨਿੱਕ-ਸੁੱਕ

(A miscellany–horizons of half-turn tilings, hobson's choice, garlands, fuchsian and kleinian actions, etc.–for my five part paper starting with khayyam.)

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On page 380 of Jordan's Traité (1870) is "Theorem. *Solving the general equation* $X = x^q + ax^{q-1} + \cdots = 0$ *reduces, after adjoining some numbers depending only on* q, *to the equation* E *which gives bisection of periods of hyperelliptic functions formed with the square root of* X:- Indeed, the roots of E, being monodromic functions of those of X, can only contain in their expressions such constants. Let us adjoin these irrationals: the roots of E will become rational functions of those of X. Let us adjoin these to the equation X: the group of this last equation will be reduced to that of substitutions leaving invariant all the roots of E, that is to say, the sole substitution 1."

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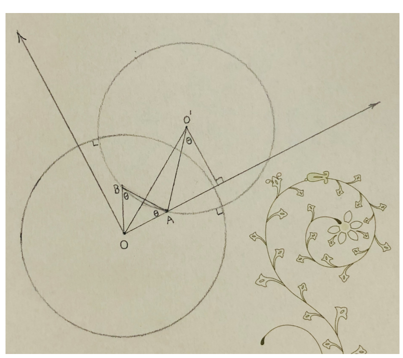
The *horizon* of a cardinality n set $\odot$ of *roots* on a circle of radius r bounds the region till which their *angle-sum* 2π *half-turns tiling* propagates. For $n = 2$ we know this region is the containing 2-sphere of radius r; for $n = 3$ one of a bigger radius $R(\odot)$; for $n = 4$ that plane, so horizon is the point ∞; and if $n > 4$ horizon bounds a containing concentric open disk of radius $R(\odot)$ depending on the roots; but, *is there a nice formula for* $R(\odot)$?

If A_1, A_2 and O are not on a line, there is a unique circle through A_1, A_2 normal to the circle of radius R around O, its centre O' the point on the right bisector of A_1A_2 such that $(OO')^2 = R^2 + (O'A_i)^2$; further,

if $OA_1 = OA_2 = r$ *and* $\angle A_1OA_2 = \phi$ *the tangents at* A_i *to the normal circle make angle* $\theta = \tan^{-1}[\frac{R^2-r^2}{R^2+r^2}\cot\frac{\phi}{2}]$ *with* OA_i (we check $\theta(R)$ increases strictly from 0 towards $\frac{\pi-\phi}{2}$ as R goes from r to ∞) :- With respect to axes along and perpendicular to OA_1—note $A_1 = A, A_2 = B$ in figure—$A_1 = (r, 0)$ and $O' = (t\cos\frac{\phi}{2}, t\sin\frac{\phi}{2})$ where $t^2 = R^2 + (t\cos\frac{\phi}{2} - r)^2 + (t\sin\frac{\phi}{2})^2$, so $t = \frac{R^2+r^2}{2r\cos\frac{\phi}{2}}$; which shows $\tan\theta = \frac{t\cos\frac{\phi}{2}-r}{t\sin\frac{\phi}{2}} = \frac{R^2-r^2}{R^2+r^2}\cot\frac{\phi}{2}$. $\square$ So,



the curved n-gon with consecutive vertices $z_j = re^{\mathbf{i}\phi_j} \in \mathbb{C}$ and sides normal to $|z| = R$ has angle-sum $\Theta = \sum_j 2\tan^{-1}[\frac{R^2-r^2}{R^2+r^2}\cot\frac{\phi_{j+1}-\phi_j}{2}]$ $\forall R \geq r$:- curved angle at z_j is $\tan^{-1}[\frac{R^2-r^2}{R^2+r^2}\cot\frac{\phi_{j+1}-\phi_j}{2}] + \tan^{-1}[\frac{R^2-r^2}{R^2+r^2}\cot\frac{\phi_j-\phi_{j-1}}{2}]$. □

Cotangent decreasing on $(0,\pi)$ with $\cot(\frac{\pi}{2}+\alpha) = -\cot(\frac{\pi}{2}-\alpha)$ checks above *even if one arc z_jz_{j+1} is a semicircle or bigger*, e.g., for $n = 2$ both curved angles are 0, but for $n > 2$ all increase continuously from 0 towards the angles of the euclidean n-gon as R goes from r to ∞. The *parallel postulate* of Euclid amounts to saying that when we describe any euclidean n-gon in the positive direction total angle we turn is 2π, i.e., the sum of internal angles is $(n-2)\pi$, the limit of the strictly increasing curved angle sum $\Theta(R)$ as R goes to ∞. Hence, *for $n > 4$, there is a unique value of R—the aforementioned horizon $R(z_1, \ldots, z_n)$—such that $\Theta(R) = 2\pi$.*

When $0 < R < r$ the above factor $\frac{R^2-r^2}{R^2+r^2}$ increases from -1 to 0, so for each degree $n > 4$ equation $\odot$ with distinct roots of equal length there also exists a unique such value at which the sum of the now negative curved angles becomes -2π. That is, the complementary tile has sum of internal angles 2π, and starting with this half-turns generate: *the tiling of a closed disk complement in $\widehat{\mathbb{C}} = \mathbb{C} \cup \infty$ obtained by reflection in the circle $|z| = r$.* Also, for each degree four equation with distinct roots of equal length, there is this reflected tiling by mobius half-turns of $\widehat{\mathbb{C}} \setminus 0$.

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Half-turns 'complexify' linear reflections, e.g., that about the midpoint of a circular arc joining points z_i and z_{i+1} of $|z| = r$ and normal to $|z| = R$ is the composition of reflections in the line through origin and midpoint and in the circle of the arc (composition in either order because these mirrors cut normally, otherwise inverse is different). The (möbius or) *baby group* of the extended plane is generated by reflections in all its circles, and has as its component of identity all orientation preserving elements, so is generated by compositions of pairs of reflections : if this pair of circular mirrors intersects in two points we obtain a baby *turn* about and by twice the angle of these intersections (of turns only those preserving the round metric are *rotations* and make a compact subgroup $SO(3)$ to which this component retracts); if mirrors intersect in just one point we'll call them *translations;* and if they don't intersect *homotheties.* The other component of the baby group consists of all orientation reversing elements, i.e., compositions of an odd number of reflections.

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Complex inverse $z \mapsto \frac{1}{z}$ gives a half-turn of $\widehat{\mathbb{C}}$ about ± 1, being the reflection $z \mapsto \overline{z}$ in the real axis followed by reflection $z \mapsto \frac{1}{\overline{z}}$ in the circle $|z| = 1$, and these mirrors cut normally in these two points. More generally, conjugation followed by reflection $\frac{z-a}{r} \mapsto \frac{r}{\overline{z}-\overline{a}}$ in circle $|z-a| = r$ gives $z \mapsto \frac{r^2}{z-\overline{a}} + a = \frac{az+r^2-|a|^2}{z-\overline{a}}$ which has—solve $z^2 - z(a+\overline{a}) - r^2 + |a|^2 = 0$—*fixed points* $a_1 \pm \sqrt{r^2 - a_2^2}$ where a_1 and a_2 are the real and imaginary parts of a. If real and distinct $\widehat{\mathbb{C}}$ turns about them moving all circles through them by same angle—see Coxeter pages 84-89—but preserves circles of the orthogonal family; if equal it translates or pushes circles tangent to the real axis at this point and preserves the orthogonal circles; if complex the fixed points are *source and sink* of this homothety preserving circles through them and pushing the orthogonal family.

Mutiplication $z \mapsto az$, a bijection of $\widehat{\mathbb{C}}$ for $a \neq 0$, is identity if $a = 1$, otherwise has only fixed points 0 and ∞, but may neither preserves circles through them, i.e., lines through origin, nor the orthogonal circles with origin centre. However multiplication by $r \neq 1$ positive real is a homothety, reflection in $|z| = 1$ followed by reflection in distinct circle $|z| = \sqrt{r}$, with ∞ as source or sink depending on $0 < r < 1$ or $r > 1$; and by $e^{\mathrm{i}\theta}, \theta \neq 0$, being reflection $z \mapsto \overline{z}$ in $\mathbb{R}$ followed by reflection in distinct line through 0 and $e^{\mathrm{i}\theta/2}$ is a turn of $\widehat{\mathbb{C}}$ about $\{0, \infty\}$ by angle θ, e.g., $z \mapsto -z$ is a half-turn.

Addition $z \mapsto z + a$ is identity if $a = 0$, otherwise this bijection of $\widehat{\mathbb{C}}$ has sole fixed point ∞ and is a translation, the composition of reflections in the lines perpendicular to $0a$ through 0 and $a/2$, in this order.

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Thus the orientation preserving component of the baby group of the extended plane consists of all bijections arising from addition, multiplication and inverse of complex numbers. So these planar numbers are *convenient in dimension two,* but—recall the baby playing with blocks—there is nothing now like the intimacy between counting and segments; besides, for the baby group of all compositions of reflections in all codimension one spheres of extended n-space, we have to live without any such convenience, once $n \geq 3$.

For example, since all orientation preserving baby bijections of the extended plane $\widehat{\mathbb{C}}$ are *all maps* $z \mapsto \frac{\alpha z+\beta}{\gamma z+\delta}$ *with coefficients such that* $\alpha\delta - \beta\gamma$ *is nonzero*—if need be we can make this determinant one—non-identity maps have *at most two fixed points*—given by $\gamma z^2 + (\delta - \alpha)z - \beta = 0$—so at most just one *mapping three points to three other,* and since $z \mapsto \frac{z-z_1}{z_2-z_1}\frac{z_3-z_2}{z_3-z}$ takes (z_1, z_2, z_3) to $(0, 1, \infty)$, this unique map exists, etc.

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Recalling about half-turn tilings, euclidean twice branched and unbranched with double valence; for any quadrilateral unbranched, degeneration though; for curved geometry unbranched with multiple valences, branched with any n-gons, fractional values of inverse function branched; straightening geometry, a suitable figure seeing herself multiple times, central reflections now, light, halo, relativity. After straightening the made tiling by that radial self-homeomorphism of the disk of radius R and using the new cayley r' we'll put $c = R/r'$; as is, in this möbius or conformal picture we'll use letter u. Maybe it is best to start from all $|z_i| = 1$ only because expression below obtainable from it obviously has scaling interpretation for $|z_i| = r$, but note excluded point 0 is all important, we don't translate it for instance, and it is schlicht mapping of open unit disk preserving its origin that we'll be constructing as $\odot$ moves more generally over the entire complex swallowtail.

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Further, $\frac{\Theta}{2} = \sum_j \tan^{-1}[\frac{R^2-r^2}{R^2+r^2}.\mathbf{i}\frac{z_{j+1}+z_j}{z_{j+1}-z_j}]$:- 'adjacent' $\frac{z_{j+1}+z_j}{2}$rotated by 90° has direction (or its negative, if $\phi_{j+1} - \phi_j > \pi$) of 'opposite' $\frac{z_{j+1}-z_j}{2}$, so 'cot = adjacent/opposite' of $\frac{\phi_{j+1}-\phi_j}{2}$ is equal to $\mathbf{i}\frac{z_{j+1}+z_j}{z_{j+1}-z_j}$.□ So far tan was restricted to $(-\frac{\pi}{2}, +\frac{\pi}{2})$ and $\frac{R^2-r^2}{R^2+r}$ to $0 < r \leq R < \infty$, but right hand side is a *multi-valued meromorphic extension of* $\frac{\Theta}{2}$ for any cyclically ordered n distinct $z_j \in \mathbb{C} \setminus 0$, and any complex number—in above case $\frac{R^2-r^2}{R^2+r^2} \in \mathbb{R}$—or *hobson's choice* $z \in \mathbb{C}$, which we'll tie to a general cyclic sequence z_j below.

Only their ratios $u_j = \frac{z_{j+1}}{z_j} \in \mathbb{C} \setminus \{0, 1\}$ enter above, but they fix the roots z_j 'if separated enough', because we can write all others in terms of say z_1, and the nonzero constant coefficient of $\odot$ gives $\prod_j z_j$ (but all elementary symmetric functions of $u_1, ..., u_n$ are not symmetric functions of $z_1, ..., z_n$). So the *degree* k *elementary symmetric functions* e_k *of* $w_j = \mathbf{i}\frac{z_{j+1}+z_j}{z_{j+1}-z_j}$ are 'sort of' known even from the coefficients of $\odot$. However like Umemura we won't go into separation, indeed for practical purposes there are well-known methods for approximating roots of any given equation to any accuracy.

The unknown is $c = \frac{R}{r}$ making $\Theta = 2\pi$, from which we can then construct by ruler and compass the entire curved half-turn tiling, for the case all $|z_j| = r$. *For the general case, c is an unknown nonzero complex number,* and let's use $z = \frac{c^2-1}{c^2+1}$.[1] With these abbreviations $\frac{\Theta}{2} = \sum_j \tan^{-1}[zw_j]$ which gives—cf. Hobson's Trigonometry (1891), §§49, 187—on taking tan of both sides

$$\tan(\Theta/2) = \frac{e_1 z - e_3 z^3 + e_5 z^5 - \cdots}{1 - e_2 z^2 + e_4 z^4 - \cdots}$$

So *setting above numerator equal to zero*– essentially a lower degree equation constructed from the coefficients of $\odot$ – gives *all circles of radii making angle-sum a multiple of* 2π *at most* $(n-2)\pi$, horizon being smallest; for $n = 4$ horizon becomes point $\{\infty\}$; but for $n = 3$ no horizon, the bulging external angle-sum 2π tile births three more completing a tetrahedral tiling of $\widehat{\mathbb{C}}$ and these half turns complete a klein four group; while the two-tiled $n = 2$ school case is its limit as one of the three roots proceeds to infinity. Thus known stuff for $n = 2, 3$ and 4 including Hermite's warm-up elliptic method for $n = 4$ should also be covered. The solving multiperiodic meromorphic functions (even the circular functions used in above calculation)—giving the quotient $\widehat{\mathbb{C}}$, by dividing by the full n half-turns group, and a *surface of genus* $[\frac{n}{2}]-1$ obtained by dividing only by its freely acting index two subgroup, of our tiling—can all be constructed by averaging *à la valentine day 1881 note of Poincaré*, and its sequel of a week later, the rational function coming from $\odot$. Solving the fuchsian differential equation tied to the latter is equivalent to inverting Legendre's indefinite hyperelliptic integral, so

[1] Or $c^2 = \frac{z+1}{z-1}$ can be unknown, likewise coefficients of equation set below can be written using ratios u_j instead of w_j, the point is our argument shows that, when $|z_j|$ are equal and in circular order, this equation has distinct real positive roots c^2.

the periods of this solution or integral over the homologically non-trivial loops of that surface give us the translations of the index two group having the union of any two adjacent tiles as a fundamental region.

For $n = 3$ calculations will be akin to that paper with Zoltek, and for $n = 4$ taking limit over disks of increasing radius gives required elliptic function from same theta function recipe of Poincaré.

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There are also reciprocal tilings under $z \mapsto \frac{r^2}{z}$ for all $n \geq 4$, for $n = 4$ this is a cuved tiling of $\widehat{\mathbb{C}} \setminus 0$ instead of straight rooftop tiling of $\widehat{\mathbb{C}} \setminus \infty = \mathbb{C}$, the point horizon 0 instead of ∞, but for $n = 2$ and 3 there is no real positive horizon, but note zero multiple of 2π so far being thrown out gives the horizon-less two tiles filling $\widehat{\mathbb{C}}$ for $n = 2$, for $n = 3$ the imaginary pair of roots of $z^2 = e_1/e_3$ – note now right side is a symmetric function of the roots of the cubic $\odot$ so can be written without solving it – gives us the two associated horizon-less tetrahedral tilings of the riemann sphere visualized in the beginning as round, but can also be stereographed and imagined as two horizonless tilings of $\widehat{\mathbb{C}}$.

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For any $n = 4$ distinct roots of unit length there is a 2π angle sum tiling of $\widehat{\mathbb{C}} \setminus c$ for each c not on the unit circle. When ∞ the rooftop tiling, when 0 its reflection – not complex inversion $1/z$, a half-turn with fixed points ± 1, because it makes roots complex conjugates – in unit circle, now arcs are from circle through 0; likewise all circles through c will gives tiling mentioned. Instead of unit here we can take any circle, even a straight line, so above applies to all roots real. As noted before when roots wander off circle (holomorphically, but lets look first at baby or mobius transformations, so new roots also concyclic) rooftop tiling collapses whenever the four become colllinear. But then one of the above, holomorphically equivalent if c stays on the same side of line (but even then mobius-distinct) tilings for the collinear roots can be used; so making a preliminary mobius transformation extends method. Also note for $n \geq 4$ that, *there are permutations* (other than the natural orders on circle) *not realized by any circularly curved tile,* e.g, if we want curved edges ± 1 and $\pm i$ they will intersect irrespective of the radiii these two circular arcs have.

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One of the key things below is what that complex extension of the horizon, for case $|z_i| = r$ in circular order, or hobson's choice c means? For any cyclically ordered n distinct complex nonzero numbers $z_1, z_2, \ldots, z_n, z_1$ and each choice of points on the right bisectors of the n segments $z_j z_{j+1}$ we have n circles passing respectively through $\{z_j, z_{j+1}\}$. Is this hobson's choice dictating perhaps an extended choice of these centres for which the group acting by the n extensions of those half-turns (composition of reflections in the bisector and the circle) though now no longer fuchsian, is still acting discontinuously on an open subset of $\widehat{\mathbb{C}}$? Or per jargon is still a kleinian group? If so, we would have an attached automorphic, or per Poincaré's original paper a kleinian function, extending the full galois monodromy of the swallowtail.

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Half-turns of $\odot$ *with roots* $z_j = re^{\mathbf{i}\phi_j}$ *about midpoints of arcs* $z_j z_{j+1}$ *normal to* $|z| = R$:- Let arc subtend angle 2ϕ at origin, have centre a and radius ρ; since $\rho^2 = (|a| - r\cos\phi)^2 + (r\sin\phi)^2$ is equal to $|a|^2 - R^2$ we get $|a| = \frac{R^2+r^2}{2r\cos\phi}$; so $z_j z_{j+1}$ has $a = \frac{R^2+r^2}{|z_j+z_{j+1}|}\frac{z_j+z_{j+1}}{|z_j+z_{j+1}|}$ on the line $z = te^{\mathbf{i}\frac{\phi_j+\phi_{j+1}}{2}} = t\frac{z_j+z_{j+1}}{|z_j+z_{j+1}|}$ through 0 and its midpoint; the required half-turn about it is the reflection $z \mapsto \overline{z}e^{\mathbf{i}(\phi_j+\phi_{j+1})}$ in line followed by the reflection $\frac{z-a}{\rho} \mapsto \frac{\rho}{\overline{z}-\overline{a}}$ in circle; which works out to be $z \mapsto \frac{(R^2+r^2)z - R^2(z_j+z_{j+1})}{r^2(z_j^{-1}+z_{j+1}^{-1})z - (R^2+r^2)} = \frac{(c^2+1)z - c^2(z_j+z_{j+1})}{(z_j^{-1}+z_{j+1}^{-1})z - (c^2+1)}$, $c = \frac{R}{r}$. $\square$

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In the classical limit $R \to \infty$ this half-turn $z \mapsto -z + z_j + z_{j+1}$ about $\frac{z_j+z_{j+1}}{2}$ followed by $z \mapsto -z + z_k + z_{k+1}$ about $\frac{z_k+z_{k+1}}{2}$ gives the euclidean translation $z \mapsto z - z_j - z_{j+1} + z_k + z_{k+1}$. For the general relativistic case $R < \infty$ this gives $z \mapsto \frac{[(c^2+1)^2 - c^2(z_k+z_{k+1})(z_j^{-1}+z_{j+1}^{-1})]z + [(c^2+1)c^2(z_j+z_{j+1}-z_k-z_{k+1})]}{[(c^2+1)(z_k^{-1}+z_{k+1}^{-1}-z_j^{-1}-z_{k+1}^{-1})]z + [(c^2+1)^2 - c^2(z_k+z_{k+1})(z_j^{-1}+z_{j+1}^{-1})]}$, a horror, so we'll use another tack to understand these.

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Even for c *complex the formulas above give* n *involutions,* viz., möbius turns through π of the sphere $\widehat{\mathbb{C}}$ about two fixed points; so preserve all circles through them, in particular their *isometric circles* having the segment joining the fixed points as a diameter. For the seminal case $|z_j| = r$ in circular order and c real and positive these circles pass through the pairs $\{z_j, z_{j+1}\}$, and for hobson's choice of c, the sum of the succesive angles between them is 2π, so they enclose a fundamental region or a tile of a *fuchsian* subgroup. For any cyclic sequence z_j of n nonzero complex numbers, these isometric circles usually separate these pairs of points, but it would seem our extended hobson's choice of c, would ensure the above angle sum is still 2π, but these n arcs can now intersect, we'll need to throw in isometric circles $|\frac{dg}{dz}| = 1$ of some compositions g too, to make a fundamental region or tile of a *kleinian* subgroup or tiling, with *horizon probably no worse than a closed smooth curve* ? Anyway, discontinuity alone will suffice by Poincaré averaging to still associate *automorphic,* i.e., meromorphic functions periodic with respect to this group action.

Poincaré loved *extended baby actions on upper half space, dihedral angles, and fundamental polyhedrons,* with above polygons their traces made in pairs on its bottom $\widehat{\mathbb{C}}$, e.g., a fuchsian subgroup tiles the two open disks complementing a circular horizon. This suggests that for a fully algebraic (as against al-jabric!) grasp of a complex horizon c, *we should perhaps curl up* $\widehat{\mathbb{C}}$ *into a round* 2*-sphere,* and seek a higher dimensional construction like the one we used to prove the existence of a real horizon in the seminal case.

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So, with any cyclic sequence z_j of distinct nonzero complex numbers, and any $c \neq 0$, comes a cyclic sequence g_j of *involutions* $z \mapsto \frac{(c^2+1)z-c^2(z_j+z_{j+1})}{(z_j^{-1}+z_{j+1}^{-1})z-(c^2+1)}$, which switch z_j and z_{j+1}, and ∞ with their *poles* $\frac{c^2+1}{z_j^{-1}+z_{j+1}^{-1}}$. We note also that, *any möbius bijection switching two points is necessarily an involution,* i.e., of order two :- if $z \mapsto z'$ switches z_1 and z_2 and w is a third point equality of cross ratios gives $\frac{(z-z_2)(w-z_1)}{(z-w)(z_2-z_1)} = \frac{(z'-z_1)(w'-z_2)}{(z'-w')(z_1-z_2)}$, and using a mobius transformation we can assume as well $w' = \infty$ when $\frac{(z-z_2)(w-z_1)}{(z-w)(z_2-z_1)} = \frac{(z'-z_1)}{-(z_1-z_2)}$, i.e., $z' = \frac{(z-z_2)(w-z_1)}{(z-w)} + z_1$ which gives $\infty' = \frac{w-z_1}{-1} + z_1 = w.$ □

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Over a real swallowtail, space of equations with all n roots real and distinct, the *covering space* of all sequences of these roots has $n!$ components, there is no monodromy. While if we allow a root at infinity–likewise if roots are constrained to an extended line or circle–these components reduce to $(n-1)!$, closed paths back to an equation lift to give cyclic permutations. But full *galois monodromy* kicks in for equations with roots distinct and in a connected open subset of the plane:- the two dimensional room now available allows us to easily make a path joining any pair of equations, with any given total orders on their roots, such that a lifted path joins these permutations. □

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Similarly the space of all cyclic sequences z_j of n distinct nonzero complex numbers is path connected as are its open subsets obtained by deleting a subset of codimension two or more, e.g., $z_j = -z_{j+1}$ for some j. *It seems likely that not only hobson's choice of c—making angle sum 2π, i.e., $\prod_j g_j = id$—but the discontinuous nature of the action generated by our n involutions g_j also spreads analytically from all $|z_j| = r$ in circular order to this huge connected space of cyclic sequences.* Discontinuity of action means it can be visualized by a tiling of an open subset of $\widehat{\mathbb{C}}$. The tiles which Poincaré made had piecewise circular boundaries, soon after his method was honed to *a fixed recipe using only isometric circles of all compositions g.* This strongly suggests our conjecture because the abstract group remains the same, generated by n involutions whose product is the identity, only its action dictated by the cyclic sequence z_j and c changes. Notably the n circular arcs, preserved by g_j on which z_j, z_{j+1} lie, can now have other intersections, but merrily ignoring these we can build by *grecian origami* a tiling too on a multi-sheeted riemann surface topologically an open ball. We suspect our kleinian action is *quasi-fuchsian,* with the horizon of the tiled open subset of $\widehat{\mathbb{C}}$ always a smooth jordan curve.

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Our garland of n involutions—we'll assume all with pole—defines a *garland of n circles, the jth being that through points z_j and z_{j+1}, which are interchanged by* $z \mapsto \frac{(c^2+1)z-c^2(z_j+z_{j+1})}{(z_j^{-1}+z_{j+1}^{-1})z-(c^2+1)}$*, and a fixed point* $\frac{c^2+1}{z_j^{-1}+z_{j+1}^{-1}} \pm \frac{\sqrt{c^4-c^2(z_j z_{j+1}^{-1}+z_{j+1}^{-1}z_J)+1}}{z_j-1+z_{j+1}^{-1}}$. Further, *which of the two fixed points we take doesn't matter, we get same circle*:- We can use a translation of $\mathbb{C}$, since it keeps ∞ fixed, to first make one of the interchanged points 0, and then a rotation and homothety, since they keep both 0 and ∞ fixed, to also make pole 1. Thus it suffices to check for $z \mapsto \frac{z-b}{z-1}$ which interchanges 0 and b that these two points, and its fixed points given by $z^2-2z+b=0$, i.e., $z = 1 \pm \sqrt{1-b}$, are concyclic. Now, *four complex numbers lie on a circle of $\widehat{\mathbb{C}}$ if and only if their cross ratios are real,* this because these are möbius invariant, and we can make any circle the extended real line. Using u for one of the square roots of $1-b$ our four points in order $0, 1-u, 1-u^2, 1+u$ have cross ratio $\frac{2u}{u+u^2} \cdot \frac{1-u^2}{1-u} = 2$, so they are concyclic. $\square$

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Above cross ratio 2 means in mob̆ius sense fixed points of involution still bisect the interchanged pair of points. Like in the prototypical case when all the z_i were at the same distance r from the origin, and cyclic order was the natural one prescribed on this circle by the complex structure. In this case the bisectors of all $z_j z_{j+1}$ were concurrent, which is no longer the case; and c was the same positive real number; if less than one all circles intersected normally a smaller concurrent circle, and the z_j appeared as the intersection away, resp. neaer from the origin of successive circles depending on whether c was smaller or bigger than one, if equal to 1 of course they were tamgent. Using the intermediate value theorem we had found for given z_j a unique c bigger and smaller than 1 for which the angle sum was exactly 2π. And the hobson equation solving which this pair of c's could be found. Note in this seminal case the poles of the involutions, the half-turns now, were exactly the centres of the circles, and fixed points the antipodal pair exactly bisecting them. *The hobson equation is unchanged if all z_j are multiplied by the same nonzero complex number t* :- because the ratios $\frac{z_{j+1}+z_j}{z_{j+1}-z_j}$ remain unchanged and the coefficients e_k entering into this equation remain the same.$\square$ So as we algebraically continue this pair of roots–which in this seminal case are positive real–this invariance is to be kept in mind: the cyclicalyl ordered sequence of distinct complex nonzero numbers z_j matters only upto a nonzero complex multiple. For the seminal case this invariance is also clear: multplying by a positive real is a homothety, by a complex number of absolute value one a rotation, by any a combination–a spiralling homothety. For the general case when almost all the intuitive ingredients of this picture evaporate, only the enigmatic hobson's equation with continuation into $\mathbb{C}$ of the seminal rea root pairs is teasing us, we want an associated equally lucid picture. A garland of circles with poles of associated involutions prescribed by the same complex number c is our tentive in this direction. Note that now as in the seminal case multiplying all z_j by same t multiplies poles too by that t, and the involutions just change as prescribed by this piralling homothety. This blown up garland has same angle sum. *An example of a non-garland is provided by a non trivial euclidean translation of the seminal garlands.* for we can't obviously find a constant complex number C such that $\frac{C^2+1}{z_{j+1}+a^{-1}+(z_j+a)^{-1}}$ equals $\frac{C^2+1}{z_{j+1}{}^{-1}+(z_j)^{-1}}+a$ for all j when the translation a is nonzero.

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The angle of $\frac{(z_{j-1}-z_j)^2}{(z_{j+1}-z_j)^2}.\frac{c^2-z_j^{-1}z_{j+1}}{c^2-z_{j+1}^{-1}z_j}.\frac{c^2-z_{j-1}^{-1}z_j}{c^2-z_j^{-1}z_{j-1}}$ *equals twice the angle at the corner* z_j *of our garland* :- The latter angle, between the jth and $(j-1)$th circle at z_j, equals that at their other intersection z_j^*, which after $z \mapsto \frac{1}{z-z_j}$ becomes the angle between two lines through $\frac{1}{z_j^*-z_j}$. If $\frac{pz-b}{z-p}$ denotes the jth involution after $z \mapsto z-z_j$, then $p=\frac{c^2+1}{z_j^{-1}+z_{j+1}^{-1}}-z_j$ is the translated pole, 0 and $\frac{b}{p}=z_{j+1}-z_j$ are interchanged, while the fixed points, the roots of $z^2-2pz+b=0$, are now $p \pm \sqrt{p^2-b}$. Their reciprocals $\frac{1}{p\pm\sqrt{p^2-b}}$ have mid-point $\frac{p}{b}=\frac{1}{z_{j+1}-z_j}$, and their difference $\frac{-2\sqrt{p^2-b}}{b}$, so $\frac{\sqrt{p^2-b}}{b}$, gives direction of this line. Its square $\frac{p^2-b}{b^2}=\frac{p}{b}[\frac{p}{b}-\frac{1}{p}]=\frac{1}{z_{j+1}-z_j}[\frac{1}{z_{j+1}-z_j}-\frac{z_j^{-1}+z_{j+1}^{-1}}{c^2+1-z_j(z_j^{-1}+z_{j+1}^{-1})}]=\frac{1}{(z_{j+1}-z_j)^2}.\frac{c^2-z_j^{-1}z_{j+1}}{c^2-z_{j+1}^{-1}z_j}$. The displayed complex number is this, divided by its analogue for the second line, for which we only need to replace z_{j+1} by z_{j-1}. □

Corollary: *the angle of* $\prod_j \frac{c^2-z_j^{-1}z_{j+1}}{c^2-z_{j+1}^{-1}z_j}$ *equals the sum of the angles of our garland* :- for its square is the product of the above n numbers. □

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The product above equals $\prod_j \frac{p_j-z_{j+1}}{p_j-z_j}=\prod_j \frac{z_j-p_{j-1}}{z_j-p_j}$ *where* p_j *denotes the pole of the* j*th involution* :- use $p_j=\frac{c^2+1}{z_j^{-1}+z_{j+1}^{-1}}$. □ Hence, its absolute value gives us the *alternating product of segments* $\prod_j \frac{p_j z_{j+1}}{z_j p_j}$, and we see, *the sum of the angles of a garland coincides with the cyclic angle sums* $\sum_j \angle z_j p_j z_{j+1}=\sum_j \angle p_j z_j p_{j-1}$:- use $\frac{p_j-z_{j+1}}{p_j-z_j}=\frac{p_j z_{j+1}}{z_j p_j}e^{\mathbf{i}\angle z_j p_j z_{j+1}}$, etc. □ For the case all $|z_j|=r$ in circular order, each $\angle p_j z_j p_{j-1}$ is equal to the angle of the garland at z_j, because poles were now the centers, and radii are normal to tangents : this equality of individual angles is *not* valid in general, only their cyclic sums coincide.

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Hence, *the sum of the angles of our garland is a multiple of* 2π *if and only if* $\tan(\sum_j \frac{1}{2\mathbf{i}}\log\frac{c^2-z_j^{-1}z_{j+1}}{c^2-z_{j+1}^{-1}z_j})=0$:- half the angle of $\prod_j \frac{c^2-z_j^{-1}z_{j+1}}{c^2-z_{j+1}^{-1}z_j}$ equals the real part of its $\frac{1}{2\mathbf{i}}$ log, and is a multiple of π if and only if its tan is zero, but complex tangent function has only real zeros. □

Further, *the above equation can be rewritten as a polynomial equation for the unknown* $c^2 \in \mathbb{C}$ *with coefficients depending on the successive ratios* $u_j = \frac{z_{j+1}}{z_j}$ *of the cyclic sequence* z_j, which, for the case all $|z_j| = r$ in circular order, must coincide with the equation we had obtained before :- $\tan(\frac{1}{2\mathbf{i}} \log w) = \mathbf{i}\frac{1-w}{1+w}$ gives $\tan(\frac{1}{2\mathbf{i}} \log \frac{c^2 - z_j^{-1} z_{j+1}}{c^2 - z_{j+1}^{-1} z_j}) = \mathbf{i}\frac{z_j^{-1} z_{j+1} - z_{j+1}^{-1} z_j}{2c^2 - z_j^{-1} z_{j+1} - z_{j+1}^{-1} z_j} := Q_j$ say, so the addition formula for tan makes our equation $\sum_m (-1)^m E_{2m+1}(Q_j) = 0$ where $E_t(\cdot)$ denotes the degree t elementary symmetric function of n quantities. Note $Q_j \neq 0$ iff $u_j \neq \pm 1$ and Q_j^{-1} is of degree 1 in c^2. Multiplying by $\prod_j Q_j^{-1}$ gives a polynomial equation $\sum_m (-1)^m E_{n-2m-1}(Q_j^{-1}) = 0$ *of degree less than* n in the unknown c^2. For the case all $|z_j| = r$ in circular order, it must coincide with the equation before, because it has the same roots. □

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Let's review the case all $|z_j| = r$ *in circular order,* when poles are centers, so radii $z_j p_j$ and $z_j p_{j-1}$ are normal to the tangents to the jth and $(j-1)$th circles at z_j, so angle of the garland at z_j equals $\angle p_{j-1} z_j p_j$, i.e., the angle of $\frac{p_j - z_j}{p_{j-1} - z_j}$. We had essentially calculated this before as the angle from $z_j p_{j-1}$ to the *ray from origin through* z_j followed by the angle from this ray to $z_j p_j$, i.e., angle of $\frac{z_j}{p_{j-1} - z_j}$ plus angle of $\frac{p_j - z_j}{z_j}$. With reference to that figure the latter angle is $90° - \theta_j$, so $\tan \theta_j$ is equal to the cotangent of the angle of $\frac{p_j - z_j}{z_j} = \frac{c^2 - z_{j+1}^{-1} z_j}{1 + z_{j+1}^{-1} z_j} = \frac{c^2 - e^{-\mathbf{i}\phi}}{1 + e^{-\mathbf{i}\phi}} = \frac{c^2 - \cos\phi + \mathbf{i}\sin\phi}{1 + \cos\phi - \mathbf{i}\sin\phi} = \frac{(c^2-1)(1+\cos\phi) + \mathbf{i}(c^2+1)\sin\phi}{1+\cos\phi}$ where $\phi = \phi_{j+1} - \phi_j$ is the angle between the consecutive points z_j and z_{j+1} on $|z| = r$. So this number's real part divided by imaginary part gives $\tan\theta_j = \frac{c^2-1}{c^2+1} . \frac{1+\cos\phi}{\sin\phi} = \frac{c^2-1}{c^2+1} . \cot\frac{\phi}{2}$ as before, then we observed $\cot\frac{\phi_{j+1} - \phi_j}{2} = \mathbf{i}\frac{z_{j+1} + z_j}{z_{j+1} - z_j}$, etc., resulting in an equation that the positive real number c^2 satisfied if and only if the angle sum of this seminal garland was a multiple of π.

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Addition formula gives tan of $\angle p_{j-1} z_j p_j$, the sum of two angles with tangent $\mathbf{i}\frac{c^2-1}{c^2+1} . \frac{u_{j-1}+1}{1-u_{j-1}}$ and $\mathbf{i}\frac{c^2-1}{c^2+1} . \frac{u_j+1}{u_j-1}$, in the special case. In general $\tan \frac{1}{2}(\angle p_{j-1} z_j p_j - \mathbf{i}\log \frac{z_j p_j}{z_j p_{j-1}}) = \tan \frac{1}{2\mathbf{i}} \log(\frac{z_j p_j}{z_j p_{j-1}} e^{\mathbf{i}\angle p_{j-1} z_j p_j}) = \tan \frac{1}{2\mathbf{i}} \log(\frac{p_j - z_j}{p_{j-1} - z_j}) = \mathbf{i}\frac{p_{j-1} - p_j}{p_{j-1} + p_j - 2z_j}$, which gives $\tan(\angle p_{j-1} z_j p_j - \mathbf{i}\log \frac{z_j p_j}{z_j p_{j-1}})$, tally the two calculations.

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The question arises if complexification of the equation before tallies with the general equation derived above that the complex number c^2 satisfies if and only if the angle sum—of a garland with z_j *any* cyclic sequence of distinct nonzero complex numbers—is a multiple of π? We were before confined to very special sequences z_j, viz., with the bisectors of segments $z_j z_{j+1}$ all going through the origin, i.e., the subspace of all cyclic sequences defined by the $n - 1$ equations

$|z_1| = |z_2| = \cdots = |z_n|$. That derivation—reviewed above—had invoked these conditions, resulting perhaps in some cancellation which made the coefficients of the powers of c^2 in the actual equation very special; continuing to use these special forms as coefficients *off* this subspace seems to be not on, so the answer to the question above seems more likely to be no ...

Anyway, let's examine a bit more this, *complexification of the equation before* $\sum_m (-1)^m E_{2m+1}(\mathbf{i}\frac{c^2-1}{c^2+1}.\frac{u_j+1}{u_j-1}) = 0$, i.e., since all $(-1)^m(\mathbf{i})^{2m+1}$ being nonzero and equal cancel out, $\sum_m E_{2m+1}(\frac{c^2-1}{c^2+1}.\frac{u_j+1}{u_j-1}) = 0$, and which on multiplying by $\prod_j \frac{c^2+1}{u_j+1}$ becomes $\sum_m E_{2m+1;n-2m-1}(\frac{c^2-1}{u_j-1};\frac{c^2+1}{u_j+1}) = 0$ where $E_{u;v}(\cdot\ ;\ \cdot)$ means the sum of all products for u indices of quantities of the first type and for v other indices quantities of the second type.

As against, *the derived general equation* $\sum_m (-1)^m E_{n-2m-1}(Q_j^{-1}) = 0$, i.e., likewise, $\sum_m E_{n-2m-1}(\frac{2c^2-u_j-u_j^{-1}}{u_j-u_j^{-1}}) = 0$, but $\frac{2c^2u_j-u_j^2-1}{u_j^2-1} = \frac{c^2-1}{u_j-1} + \frac{c^2+1}{u_j+1} - 1$, so the same as $\sum_m E_{n-2m-1}(\frac{c^2-1}{u_j-1} + \frac{c^2+1}{u_j+1} - 1) = 0$, so hmm ...

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Indeed, *the complexification of the equation before is equivalent to the derived general equation* :- For all $|z_j| = r$ in circular order, poles p_i being the centers, $\sum_m E_{2m+1}(\frac{c^2-1}{c^2+1}.\frac{u_j+1}{u_j-1}) = 0$ says $\sum_j \angle p_{j-1}z_jp_j$, the angle of $\prod_j \frac{z_j-p_{j-1}}{z_j-p_j}$, is a multiple of 2π, i.e., since this cyclic product can be rewritten $\prod_j \frac{p_j-z_{j+1}}{p_j-z_j}$, that $\sum_j \angle z_jp_jz_{j+1}$ is a multiple of 2π. Referring again to the figure we see a triangle with external angle $90^\circ - \theta = \frac{1}{2}\angle z_jp_jz_{j+1} + \frac{\phi}{2}$, sum of its two other internal angles, so $\tan\frac{1}{2}\angle z_jp_jz_{j+1} = \frac{\tan(90^\circ-\ \theta)-\tan(\frac{\phi}{2})}{1+\tan(90^\circ-\ \theta)\tan(\frac{\phi}{2})}$, but $\tan(90^\circ-\theta) = \frac{c^2+1}{c^2-1}\tan(\frac{\phi}{2})$ and $\tan(\frac{\phi}{2}) = \frac{1}{\mathbf{i}}\frac{u_j-1}{u_j+1}$, which gives $\tan\frac{1}{2}\angle z_jp_jz_{j+1} = \frac{-2\mathbf{i}(u_j^2-1)}{(c^2-1)(u_j+1)^2-(c^2+1)(u_j-1)^2} = -\mathbf{i}\frac{u_j^2-1}{2c^2u_j-u_j^2-1}$, etc., to put the equation before for this special case in the exact same form as the derived general equation. □

In retrospect the key step which almost ensured this minor miracle was just writing $\cot\frac{\phi_{j+1}-\phi_j}{2} = \mathbf{i}\frac{z_{j+1}+z_j}{z_{j+1}-z_j}$ since the right side now made sense even off the subset defined by the *real* equations $|z_1| = |z_2| = \cdots = |z_n|$.

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We recall *our construction for making a circular arc $\widehat{AB}$ less and less convex with chord AB as limit* : let the given $\widehat{AB}$ be on the southerly latitude of radius r of *a varying round sphere* of radius $R \geq r$; if $R = r$ the arc is equatorial, but if $R > r$ the unique great circle through A and B cuts their latitude only in these two points; and its part to the south, projected from the north pole on the plane of our latitude, is the required less and less convex circular arc tending to the chord AB as $R \to \infty$.

This projection (being a restriction of the inversion of $\widehat{\mathbb{R}^3}$ in the sphere with centre north pole passing through our latitude) is angle preserving. So now the (acute) angle θ, between the tangent at A (or B) of our varying circular arc and the radius, is equal to *the angle at A between its longitude and the great circle through A and B.* Using coordinates in which sphere is $x^2+y^2+z^2=R^2$ with poles $(0,0,\pm R)$, its tangent plane at $A=(x_0,y_0,z_0)$ is $x_0x+y_0y+z_0z=R^2$, the longitude of A lies on the plane $-y_0x+x_0y=0$, and the great circle through A and $B=(x_1,y_1,z_1)$—here $z_1=z_0$ because points are on same latitude—lies on $(y_0z_1-z_0y_1)x+(z_0x_1-x_0z_1)y+(x_0y_1-y_0x_1)z=0$. So θ is the angle between the lines tangent to the sphere at A in these two planes.

Rotating the x and y axes we make $y_0=0$, and consider the lines through the origin parallel to these two lines, i.e., $x_0x+z_0z=0, y=0$ and $x_0x+z_0z=0, -z_0y_1x+(z_0x_1-x_0z_0)y+x_0y_1z=0$, i.e., $x=z_0t, y=0, z=-x_0t$ and $x=z_0t, y=\frac{z_0^2y_1+x_0^2y_1}{z_0x_1-x_0z_0}t, z=-x_0t$. For $t=1$ these vectors have dot product $z_0^2+x_0^2=R^2$ and lengths R and $\sqrt{R^2+\frac{R^4y_1^2}{z_0^2(x_1-x_0)^2}}$, which gives $\sec^2\theta=1+\frac{R^2y_1^2}{z_0^2(x_1-x_0)^2}$, so $\tan^2\theta=\frac{R^2y_1^2}{z_0^2(x_1-x_0)^2}=\frac{R^2}{R^2-r^2}\cot^2\frac{\phi}{2}$, where again ϕ is angle subtended by the chord AB at the centre of the circle of radius r. *So* $\tan\theta=\sqrt{\frac{R^2}{R^2-r^2}}\cot\frac{\phi}{2}$, *much like before, except the first factor is different.*

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The plan below is to associate functions to *angle sum* 2π *garlands*—their existence for each cyclic sequence of distinct nonzero complex numbers z_j is now established because of FTA applied to hobson's equation—preserved by all n involutions or else only by the index two subgroup of their even compositions. This takes us beyond the complex baby group, similarities of $\widehat{\mathbb{C}}$, these functions only *preserve similarity of the smallest planar parts,* e.g., the absolute value of the quotient of planar infinitesimals, their complex derivative, is the change of scale of infinitesimal figures. Poincaré's original averaging construction of these *periodic* - with respect to the group of all compositions of g_j or its index two subgroup - functions is still the simplest, their domain being a simply connected open subset of $\widehat{\mathbb{C}}$ on which these groups act discontinuously. Dividing it by the smaller group gives a covering of a surface of genus jumping by one as n jumps by two. Everything varying continuously as $\odot$ varies over the swallowtail, but the varying domain of definition of this varying periodic or *automorphic function* can be crazy, which is what enables us to get all possible permutations of roots à la Galois. Indeed as he saw it, not in the definition theorem corollary dry manner of texts today, though granted this formal analysis has its upsides and has given natural generalizations too. For the seminal case all $|z_j|=r$ in circular order, the fundamental region or tile bounded by the isometric circles $|\frac{dg_j}{dz}|=1$ is the same as that by the n arcs z_jz_{j+1}, and iterating under the group action gives a tiling fading away to that circular horizon of radius a real positive c. This fuchsian action becomes more and more crazy or kleinian as we go off-broadway so to speak, but all the paraphernalia we have mentioned carries along at least continuously for all cyclic sequences z_j.

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Poincaré averaged rational functions R over a complex baby subgroup G thus, $\sum_{g\in G} R(g(z))(\frac{dg}{dz})^m := \Theta(z)$ wherever the sum makes meromorphic sense : see his very first note on this subject—tome 2 of his *Oeuvres*—which appeared on Valentine's Day of 1881. Since $\Theta(g(z)) = (\frac{dg}{dz})^{-m}\Theta(z)$ the quotients F of these *theta functions* of G of order m gave him oodles of functions fully periodic with respect to the baby subgroup : $F(g(z)) = F(z)\ \forall g \in G$.

Poincaré called a subgroup represented by a tiling of an open disk *fuchsian,* classified them all (!) and proved thetas meromorphic in the open disk and the complement in $\widehat{\mathbb{C}}$ of its closure; then the much bigger class of baby subgroups G that can be seen as tilings of any open planar subset Ω, using now the adjective *kleinian,* showed his averaging made meromorphic sense in this open subset, and almost classified (!!) all such groups too.

For us G will be the baby subgroup generated by the involutions g_j of the angle sum 2π garland of some cyclic sequence $\odot$ of n nonzero distinct complex numbers z_j (or its index two subgroup of even compositions). Note the abstract group is always the same, but this 'action' $G(\odot)$ depends hugely on the cyclic sequence. When all $|z_j| = r$ in circular order we know it is fuchsian, and for the moment we'll just assume it is always kleinian.

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Any kleinian subgroup G equips the open planar *halo* Ω, till whose horizon any tiling representing it is seen to fade away, with a geometry *relativistic* in the sense of *PG&R* (2013). Indeed, if all $|z_j| = 1$ in circular order, the halo of our G is concentric of radius $c > 1$, and a radial homeomorphism keeping horizon fixed makes it the open disk geometry of that paper, so *imho, our hobson's choice c complexifies speed of light* to tackle the general case of any cyclic sequence of distinct nonzero numbers! *Otoh, hobson's choice c seems an intimate of the theta constants of Riemann* (1857)! We recall that this masterly memoir lies on a thread, after Abel and Jacobi, going back to Legendre's discovery of a charming discrete ambiguity in hyperelliptic line integrals. Galois too, after his definitive analysis of Lagrange's review of available algebraic methods, was thinking on similar lines when his life was cut short, as Jordan's exposition of galois monodromy clearly points out, so that cryptic theorem on page 380 of his *Traité* (1870) also ties into this thread, and a formula just derived by Thomae from Riemann's memoir made this explicit.

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Next on agenda : for any cyclic z_j, and its c, involutions g_j and G, if $R =$ xxx then kleinian function xxx solves the differential equation xxx, i.e., inverts the hyperelliptic 1-form xxx whose periods, i.e., values over all closed paths of surface xxx, depend only on the coefficients of $\odot$; so, but for the ambiguity of the branched double cover xxx, in Jordan's parlance bissection des racines, this is a method for exactly solving any $\odot$ (which has little to do with the well-known methods for numerically solving a given $\odot$ to any accuracy).

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The 'Four Half Turns' tilings and Jacobi's elliptic functions make the lowest degree four case of the above :- because now $c = \infty$ for any $|z_j| = r$ in circular order, so its extension hobson's choice c is also this constant, and cyclic order is unimportant : the elliptic function having zeros at vertices of these quadrilateral tiles, and simple poles at the intersections of their diagonals, inverts the elliptic integral $\int \frac{dz}{\sqrt{f(z)}}$ where the four roots of $f(z) = 0$ are the values of the function at the midpoints of the tiles. $\square$

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Hyperelliptic integrals $\int \frac{z^i dz}{\sqrt{f(z)}}$ are also what go into Umemura's general recipe; a standard basis of their periods can be found using just the coefficients provided we have *separated* the unknown roots enough and put them into any order $x_1, x_2, \ldots$; then a matrix of periods is defined using this data, and viewed as a point in a higher dimensional analogue of the half plane; and a theta function built to operate on it outputs the ratio $\frac{x_1 - x_3}{x_1 - x_2}$.

The sufficiently separated roots $x_1, \ldots, x_n$ *of* $\odot$ *are fixed by all ratios* $\frac{x_i - x_j}{x_i - x_k}$:- Since $\frac{x_i - x_j}{x_l - x_k} = \frac{x_i - x_j}{x_i - x_k} \times \frac{x_i - x_k}{x_l - x_k}$, we know any $x_i - x_j$ as a multiple of $x_1 - x_2$. Also we know which square root $\sqrt{\Delta}$ of the discriminant of $\odot$ is $\prod_{i<j}(x_i - x_j)$ if roots are sufficiently separated. So $\sqrt{\Delta}$ is a constant times a $(x_1 - x_2)^N$, so $x_1 - x_2$ is known by honing separation further if need be. Which determines all x_i precisely, because from $\odot$ we can read the root-sum. $\square$

This shows we can avoid adding some known roots before using above theta; also we recall that even the periods $\oint \frac{dz}{f(z)}$ are interesting.

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The roots of $\odot$ as values of an automorphic function on fixed points of the involutions g_j recalls again the isometric circles $|\frac{dg}{dz}| = 1$, for involutions the circles having segments joining their fixed points as diameters; and a probable *canonical tiling* of the halo $\Omega(\odot)$ using isometrical circles of all $g \in G(\odot)$ which follows the galois monodromy of the roots; but to understand our complexified speed of light $c(\odot)$ more maybe we should curl up all of $\widehat{\mathbb{C}}$ into the boundary of the open unit 3-ball and look at tilings by curved polyhedra having vertices on boundary and a dihedral angle sum 2π.

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The mere fact that Abel and Jacobi preferred to view some multiple-valued line integrals of Legendre in the opposite direction as single valued functions with periods was (imho) no big deal. Indeed soon Riemann (1851,57) was looking again at periods as values of indefinite integrals, i.e., differential 1-forms, on closed paths of the graph of their finite-valued integrand, and showed how they tied to the topology of this *riemann surface.* Later Poincaré (1895) similarly used ambiguities of multiple integrals to define what much later was reborn again as the de Rham cohomology of any smooth manifold.

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From the point of view of just calculating integrals, that glue he used to make a single riemann surface from some sheets of paper cut à la Cauchy does not add much, so to many the totally new thing in his 1851 thesis was only its blemished part, an extravagant appeal to a principle of Dirichlet to obtain the *riemann mapping theorem.* However, in the lectures on which his 1857 memoir was based, Riemann completely disengaged his surfaces from the plane, in particular, he often deemed $\widehat{\mathbb{C}}$ *via stereography as a round* 2*-sphere* in 3-space. Spherically curved polytopes with vertices on a round d-sphere and facets perpendicular to a concentric sphere are interesting even for $d > 1$. Also we note that, for any two points z_j and z_{j+1} on a round 2-sphere, there is a circle's worth of 2-spheres passing through them and perpendicular to a given concentric 2-sphere, which might be the key to a better understanding of hobson's choice, or complexified speed of light c, determined by any cyclic sequence z_j.

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Were I to teach a course on complex analysis today, I would use as text Briot and Bouquet's beautiful book of 1859. All the standard material needed for the conceptual core of the theory of doubly periodic meromorphic functions is developed, besides no modern text can quite convey the freshness of learning all this while its creators, Cauchy and Liouville—the elegance of this book probably reflects that of some lectures of the latter the two authors had attended some years before and taken notes of—were still around.

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Mumford's *Tata II* (1984) has Umemura on pp. 3.261-3.272, using a formula of Thomae (1869), reproved on its pp. 3.120-3.136, making the theorem on page 380 of Jordan's *Traité* (1870)—which shows any equation can be solved using periods of hyperelliptic integrals and modular functions—more explicit. To wit he gives, for any odd degree equation with roots x_i separated into any order, a formula for $\frac{x_1-x_3}{x_1-x_2}$ in terms of a matrix of periods acted on by a formidable theta function. So, applied to the associated equation of degree n two or three more with $x_1 = 0,\ x_2 = 1$ or $x_1 = 0,\ x_2 = 1,\ x_n = 2$, a formula for a root x_3 of any equation. Of the remarks which follow the third, alas too brief, is more conceptual and evokes the galois theory of Jordan.

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Jordan in the preface of *Traité* (1870) ties it to "les OEuvres de Galois, dont tout ceci n'est qu'un Commentaire" and dwells on Galois's ideas on *division of transcendental functions* – Gauss worked on constructibility of regular polygons, i.e., dividing a circular arc, Abel on dividing elliptical arcs – which yield that theorem on page 380 in *Livre III* of this treatise. A quick recap of congruences is *Livre I,* then the much longer *Livre II* on groups of permutations and linear substitutions. But the key definition of Galois, the monodromy of the roots—covering spaces, the natural home of normal subgroups, are almost explicit in it—comes in *Livre III,* which soon turns into a review of known transcendental methods for solving equations, and stumbles all of a sudden near its end on that decisive theorem on page 380. The strict subset called galois theory today comes mostly later in *Livre IV,* as it logically should, since it deals with obstruction theoretic problems, solvability by radicals, etc.

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