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ALGEBRAIC TOPOLOGY

Unknotting and Colouring of Polyhedra

by

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Summary. We prove that an n-dimensional simplicial complex can unknot in $\mathbf{R}^{2n+1}$ only if it has an $(n-1)$-simplex which is incident to less than $3(n+1)$ n-simplices.

Introduction. A (finite) simplicial complex K will be said to *unknot* in a piecewise linear (p.l.) space Y if any two homotopic p.l. embeddings of $X = |K|$ in Y are isotopic.

THEOREM (2.2.4). *An n-dimensional simplicial complex K^n unknots in $2n+1$-dimensional space $\mathbf{R}^{2n+1}$ only if it has an $n-1$-simplex which is incident to less than $3(n+1)$ n-simplices.*

We conjecture that a conclusion similar to that of above theorem is valid also under the hypothesis that the n-dimensional simplicial complex K^n p.l. embeds in $2n$-space $\mathbf{R}^{2n}$. For $n = 1$ this is a well-known result of chromatic graph theory going back at least to Heawood [3].

An immediate consequence (2.2.5) of the above theorem is that *if K^n unknots in $\mathbf{R}^{2n+1}$ then $3(n+1)$ colours can be assigned to the $n-1$-simplices of K^n in such a way that not all the faces of an n-simplex have the same colour.* Similarly, the conjecture made above implies a similar chromatic conclusion for a K^n embeddable in $\mathbf{R}^{2n}$.

For examples of n-dimensional polyhedra which unknot in $2n+1$-dimensional Euclidean space see Husch [4]. There it is also shown that given any n-dimensional polyhedron X^n, $n \geq 2$, one can find another, which has the same simple homotopy type as X, and which unknots in $\mathbf{R}^{2n+1}$.

2. Unknotting and colouring.

(2.1) *Each embedding $\varphi\colon X \to \mathbf{R}^m$ gives rise to a map $\varphi^*\colon X^* \to S^{m-1}$ defined by*

$$\varphi^*(x_1, x_2) = \frac{\varphi(x_1)-\varphi(x_2)}{\|\varphi(x_1)-\varphi(x_2)\|}.$$

Here X^* denotes the *deleted Cartesian product of* X, i.e. all points (x_1, x_2) of $X \times X$ such that $x_1 \neq x_2$; and S^{m-1} denotes the unit sphere of the Euclidean space $\mathbf{R}^m$. We equip X^* (resp. S^{m-1}) with the free $\mathbf{Z}_2$ action given by the fixed point free involution $(x_1, x_2) \mapsto (x_2, x_1)$ – (resp. $x \mapsto -x$). We note that φ^* is *equivariant*, i.e. commutes with these involutions. Also note that if φ_0 is isotopic to φ_1 via the isotopy φ_t, $0 \leqslant t \leqslant 1$, then φ_0^* is homotopic to φ_1^* via the homotopy of equivariant maps φ_t^*.

(2.1.1) *If* $2m > 3(n+1)$, *then* $\varphi \mapsto \varphi^*$ *sets up a bijective correspondence between isotopy classes of embeddings of* X^n *in* $\mathbf{R}^m$ *and equivariant homotopy classes of equivariant maps* $X^* \mapsto S^{m-1}$.

This is Weber's classification theorem. (See [9], Th.1 and Th.1′). An analogous classification theorem is valid, under the same dimensional restrictions, also in the smooth category. This had been established earlier by Haefliger [2].

Each $\mathbf{Z}_2$ space E associates to the two-fold covering space $X^* \to X^*/\mathbf{Z}_2$ a fibre bundle $X^* \times_{\mathbf{Z}_2} E \to X^*/\mathbf{Z}_2$ with fibre E; $X^* \times_{\mathbf{Z}_2} E$ is the quotient of $X^* \times E$ under the diagonal $\mathbf{Z}_2$ action, and the projection map is defined by $[(x_1, x_2), e] \to [x_1, x_2]$. For example, we have the *$m-1$-sphere bundle* $X^* \times_{\mathbf{Z}_2} S^{m-1}$ and the associated *bundle of integer coefficients* $X^* \times_{\mathbf{Z}_2} \pi_{m-1}(S^{m-1})$. Here the $\mathbf{Z}_2$ action on S^{m-1} and $\pi_{m-1}(S^{m-1})$ is induced by the antipodal involution. The isomorphism class of the bundle of coefficients depends only on the parity of $m-1$. For $m-1$ even we denote it by $\hat{\mathbf{Z}}$ and for $m-1$ odd one has the trivial bundle $\mathbf{Z}$.

(2.1.2) *If dim* $X \leqslant n$ *the equivariant maps* $X^* \to S^{2n}$, *are in bijective correspondence with the elements of the cohomology group* $H^{2n}(X/\mathbf{Z}_2; \hat{\mathbf{Z}})$.

As Conner and Floyd (p. 419, [1]) point out this follows immediately from the following two facts:

(a) There is a bijective correspondence between equivariant maps (resp. equivariant homotopy classes of equivariant maps) $X^* \xrightarrow{\varphi^*} S^{m-1}$ and sections (resp. homotopy classes of sections) $X^*/\mathbf{Z}_2 \xrightarrow{\hat{\varphi}} X^* \times_{\mathbf{Z}_2} S^{m-1}$ of the $m-1$-sphere bundle $X^* \times_{\mathbf{Z}_2} S^{m-1}$ – one defines $\hat{\varphi}([x_1, x_2]) = [(x_1, x_2), \varphi^*(x_1, x_2)]$

(b) Steenrod's bundle-theoretic generalization of the Hopf classification theorem (see [7], §37.5, p. 186): This tells us that the homotopy classes of sections of the $2n$-sphere bundle are in bijective correspondence with the cohomology group $H^{2n}(X^*/\mathbf{Z}_2; \hat{\mathbf{Z}})$.

(2.1.3) X^n, $n \geqslant 2$, *unknots in* $\mathbf{R}^{2n+1}$ *iff* $H^{2n}(X^*/\mathbf{Z}_2; \hat{\mathbf{Z}}) = 0$ *and thus only if* $H_{2n}(X^*/\mathbf{Z}_2; \mathbf{Z}_2) = 0$.

The first part follows immediately from (2.1.1) and (2.1.2).

The short exact sequence of bundle maps $0 \to \hat{\mathbf{Z}} \xrightarrow{\times 2} \hat{\mathbf{Z}} \to X^*/\mathbf{Z}_2 \times \mathbf{Z}_2 \to 0$ gives a short exact sequence of cochain complexes $0 \to C^*(X^*/\mathbf{Z}_2; \hat{\mathbf{Z}}) \to C^*(X^*/\mathbf{Z}_2; \hat{\mathbf{Z}}) \to C^*(X^*/\mathbf{Z}_2; \mathbf{Z}_2) \to 0$. The induced long exact cohomology sequence furnishes us with a surjection $H^{2n}(X^*/\mathbf{Z}_2;\hat{\mathbf{Z}})\to H^{2n}(X^*/\mathbf{Z}_2;\mathbf{Z}_2)$. Thus $H^{2n}(X^*/\mathbf{Z}_2;\hat{\mathbf{Z}})=0$ only if $H^{2n}(X^*/\mathbf{Z}_2;\mathbf{Z}_2)\cong H_{2n}(X^*/\mathbf{Z}_2;\mathbf{Z}_2)=0$.

(2.2) We associate to each simplicial complex K the cell complex K^* consisting of all cells $\sigma^p\times\theta^\varepsilon$, $\sigma^p\in K$, $\theta^\varepsilon\in K$, $\sigma^p\cap\theta^\varepsilon=\emptyset$. We denote the space $|K|$ of K by X. The involution of X^* preserves the subspace $|K^*|\subseteq X^*$, mapping each cell $\sigma^p\times\theta^\varepsilon$ onto $\theta^\varepsilon\times\sigma^p$. Identifying $\sigma^p\times\theta^\varepsilon$ and $\theta^\varepsilon\times\sigma^p$ under this involution we get a cell $[\sigma^p\times\theta^\varepsilon]\subseteq X^*/\mathbf{Z}_2$; these cells constitute a cell complex $K^*/\mathbf{Z}_2$.

(2.2.1) $|K^*|$ *(resp.* $|K^*/\mathbf{Z}_2|$*) is a deformation retract of* X^* *(resp.* $X^*/\mathbf{Z}_2$*)*

This simple lemma occurs in van Kampen [8] (or see Shapiro [6], Lemma 2.1, or Wu [10], Ch. 1).

If σ is an i-cell of a cell complex L, $\sigma_L(\sigma)$ will denote the number of $i+1$-cells of L which are incident to σ. We put $\delta_i(L)=\inf\{\delta_L(\sigma)\,|\,\sigma\in L,\ \dim\sigma=i\}$.

(2.2.2) *For any n-dimensional simplicial complex K with* $\delta_{n-1}(K)\geqslant n+1$, $\delta_{n-1}(K)\geqslant\delta_{2n-1}(K^*)=\delta_{2n-1}(K^*/Z_2)\geqslant\delta_{n-1}(K)-n-1$.

A $2n$-cell of K^* (resp. $K^*/\mathbf{Z}_2$) is incident to the $2n-1$-cell $\sigma^{n-1}\times\theta^n$ or $\theta^n\times\sigma^{n-1}$ (resp. $[\sigma^{n-1}\times\theta^n]$), here $\sigma^{n-1}\in K$, $\theta^n\in K$, $\theta^n\in K$, $\sigma^{n-1}\cap\theta^n=\emptyset$, iff it is of the type $\xi^n\times\theta^n$ or $\theta^n\times\xi^n$ (resp. $[\xi^n\times\theta^n]$) where $\xi^n\in K$, $\sigma^{n-1}\subseteq\xi^n$, $\xi^n\cap\theta^n=\emptyset$. Moreover, since θ^n has $n+1$ vertices, out of all n-simplices $\xi^n\supseteq\sigma^{n-1}$ there can be at most $n+1$ which are not disjoint from θ^n. Thus we have

$$\delta_K(\sigma^{n-1})\geqslant\delta_{K^*}(\sigma^{n-1}\times\theta^n)=\delta_{K^*}(\theta^n\times\sigma^{n-1})$$
$$=\delta_{K^*/\mathbf{Z}_2}([\sigma^{n-1}\times\theta^n])\geqslant\delta_K(\sigma^{n-1})-n-1.$$

This implies the required result provided that K is such that for each $\sigma^{n-1}\in K$ one can find a $\theta^n\in K$ disjoint from σ^{n-1}.

If K has an $n-1$-simplex σ^{n-1} which meets every n-simplex $\theta^n\in K$, then we must have $\delta_K(\eta^{n-1})\leqslant n$ for any $n-1$-simplex η^{n-1} with card $(\eta\cap\sigma)$ least. This follows because the new vertex, of any n-simplex which is incident to η^{n-1}, must belong to σ^{n-1}. Thus $\delta_{n-1}(K)\leqslant n$ and the result follows.

(2.2.3) *For any n-dimensional simplicial complex K with* $H_{2n}(K^*/\mathbf{Z}_2;\mathbf{Z}_2)=0$ *one has* $\delta_{n-1}(K)<3(n+1)$.

By (2.2.2) it suffices to prove $\delta_{2n-1}(K^*/Z_2)<2(n+1)$. Also we can assume that K^*/Z_2 has at least one $2n-1$-cell: otherwise we have in fact $\delta_{n-1}(K)\leqslant n$ as in the proof of (2.2.2).

We note that $\dim C_i(K^*/\mathbf{Z}_2;\mathbf{Z}_2)=$ number of i-cells of $K^*/\mathbf{Z}_2$. Since each $2n$-cell $[\sigma^n\times\theta^n]$ of $K^*/\mathbf{Z}_2$ has precisely $2(n+1)$ incident $2n-1$-cells, namely those of type $[\xi^{n-1}\times\theta^n]$, $\xi^{n-1}\subseteq\sigma^n$ and $[\sigma^n\times\zeta^{n-1}]$, $\zeta^{n-1}\subseteq\theta^n$, it follows that $\delta_{2n-1}(K^*/\mathbf{Z}_2)$. $\dim C_{2n-1}(K^*/\mathbf{Z}_2;\mathbf{Z}_2)$ is less than or equal to $2(n+1)$. $\dim C_{2n}(K^*/\mathbf{Z}_2;\mathbf{Z}_2)$. This in turn is less than $2(n+1)\cdot\dim C_{2n-1}(K^*/\mathbf{Z}_2;\mathbf{Z}_2)$ because, under the given hypotheses $H_{2n}(K^*/\mathbf{Z}_2;\mathbf{Z}_2)=0$ and $\dim C_{2n-1}(K^*/\mathbf{Z}_2;\mathbf{Z}_2)\geqslant 1$, the mod 2 boundary map $\partial\colon C_{2n}(K^*/\mathbf{Z}_2;\mathbf{Z}_2)\to C_{2n-1}(K^*/\mathbf{Z}_2;\mathbf{Z}_2)$ is injective and its image is a proper subspace of $C_{2n-1}(K^*/\mathbf{Z}_2;\, \mathbf{Z}_2)$. So we get the required estimate $\delta_{2n-1}(K^*/\mathbf{Z}_2) < (n+1)$.

(2.2.4) *If simplicial complex K^n unknots in Euclidean $2n+1$-space $\mathbf{R}^{2n+1}$, then $\delta_{n-1}(K) < 3(n+1)$.*

For $n = 1$ one has in fact $\delta_0(K^1) = 1$ because K^1 unknots in $\mathbf{R}^3$ iff it has not loops, i.e. iff it is a disjoint union of trees.

For $n \geqslant 2$, (2.1.3), (2.2.1) and (2.2.3) yield the above result.

As in [5] we define the *i-th chromatic number of* K as the least number of colours that can be assigned to the i-simplices of K in such a way that not all the i-faces of any $i+1$-simplex have the same colour. We denote this number by $c_i(K)$.

(2.2.5) *If K^n unknots in $\mathbf{R}^{2n+1}$, then $c_{n-1}(K^n) \leqslant 3(n+1)$.*

To see this we observe that if K^n unknots in $\mathbf{R}^{2n+1}$ so does any subcomplex L^n. By (2.2.4) we can find a $\sigma^{n-1} \in L^n$ incident to less than $3(n+1)$ n-simplices of L^n. Thus any good colouring of $L^n - \mathrm{St}_{L^n}\sigma^{n-1}$ can be extended to a good colouring of L^n. Proceeding step by step we can colour all of K^n in the requisite way. For $n = 1$ one has in fact $c_0(K^1) = 2$.

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