A GENERALIZED WU LEMMA

by

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§1. Introduction

If K is a (finite) simplicial complex, then its *pth deleted product* is the cell complex consisting of all cells of the type $\sigma_1 \times \ldots \times \sigma_p$, where the σ_i's are nonempty simplices of K with $\sigma_1 \cap \ldots \cap \sigma_p = \emptyset$.

Wu [ ] showed that the **homotopy types of its deleted products are topological invariants of a simplicial complex**: in fact that the above cell complex has the same homotopy type as the space obtained from the p-fold cartesian product of the space $X = |K|$ by deleting its diagonal points $(x_1, \ldots, x_p)$, $x_1 = \ldots = x_p$.

One might ask whether the homotopy type of the smaller cell complex of all cells $\sigma_1 \times \ldots \times \sigma_p$ having *pairwise disjoint* factors σ_i's is also a topological invariant of K ? And even, whether this cell complex has the same homotopy type as the *pth configuration space* of X, i.e. the subspace of the p-fold product consisting of all pairwise distinct sequences $(x_1, \ldots, x_p)$ of points of $X = |K|$?

The answer to both these questions is "no". This is most easily seen for the case when K is a closed n-simplex and $p \geq 2^{n+1}$. Now the aforementioned cell complex is empty, but if we subdivide K finely enough, then the analogous cell complex of this subdivision is not empty.

The object of this note is to show that generalizations analogous to the ones just considered do *hold provided one uses* joins *rather than products:*

In fact if the simplices of the p-fold join $K \cdot \ldots \cdot K$ are denoted $(\sigma_1, \ldots, \sigma_p)$[1], the σ_i's being (possibly empty) simplices of K, then we'll show, for each $1 \leq j \leq p$, that **the homotopy type of the subcomplex of** $K\cdot$

[1] More precisely $K \cdot \ldots \cdot K$ denotes the join of p mutually disjoint *copies* ${}^iK$ of K, and $(\sigma_1, \ldots, \sigma_p)$ stands for the *disjoint union* ${}^1\sigma_1 \cup \ldots \cup {}^p\sigma_p$, ${}^i\theta \in {}^iK$ being the copy in ${}^iK$ of any simplex $\theta \in K$: see (2.4).

... $*$ K **consisting of all j-wise disjoint[2] sequences** $(\sigma_1, \ldots, \sigma_p)$ **is a topological invariant of** K. Moreover we'll show that if a suitably defined "jth diagonal" is deleted from the space $X*\ \ldots * X$, then the same homotopy type is obtained.

Method of proof. Wu established his lemma by employing a certain cell subdivision of $K \times \ldots \times K$.[3] The use of joins clarifies his proof since this cell subdivision is the section of a pleasant simplicial subdivision of the simplicial complex $K* \ldots * K$. Moreover this latter subdivision admits natural generalizations which yield the required generalized Wu lemma.

§2. Generalized Wu Subdivisions

(2.1) As usual a **simplicial complex** K will be a finite set of finite sets such that if $\sigma \in K$ and $\theta \subseteq \sigma$, then $\theta \in K$, and we will assume the usual notations and terminology concerning simplicial complexes.

In particular we recall that a simplicial complex L is called a **stellar subdivision** of K iff it is obtainable (upto simplicial isomorphism) from the latter by a sequence of operations in each of which one **derives** some nonempty simplex σ at its **barycentre** $\hat{\sigma}$, i.e. replaces $\overline{\mathrm{St}(\sigma)}$ by $\hat{\sigma}.\partial(\sigma).\mathrm{Lk}(\sigma)$.

(2.2) It is unknown whether the binary relation of having a common stellar subdivision is transitive, and thus the same as the equivalence relation generated by these subdivisions.

On the other hand a fundamental theorem of M.H.A.Newman [ ] does identify this equivalence relation with that of being *piecewise linearly homeomorphic*, i.e. of having a common *geometric subdivision*.[4]

However, ignoring the above shortcoming (in our knowledge) of stellar subdivisions, we will keep the following treatment purely combinatorial by assuming that *all subdivisions mentioned below are stellar*, even though the definitions make sense for more general notions like geometric subdivisions.

(2.3) We recall that, if any two of the simplicial complexes L_i, where the index i runs over a finite set $\pi = \{a, b, \ldots\}$, are *disjoint*, i.e. have only the empty simplex in common, then the simplicial complex

[2] i.e. such that the intersection of any j of the σ_i's is empty.

[3] Expositions of this proof are given in Hu [ ] and in Wu's book [ ]. We note also that the proof given in Shapiro [ ] is erroneous.

[4] It is easy — cf. Hudson [ ] — to further ensure that this common geometric subdivision be a stellar subdivision of *one* of the simplicial complexes.

$\{\bigcup_{i\in\pi} \theta_i : \theta_i \in L_i\}$ is called their **join**, and is denoted by $*_{i\in\pi}(L_i)$ or $L_a * L_b * \ldots$.

By the **ith copy** ${}^iK$ of a simplicial complex K we will mean the isomorphic simplicial complex, consisting of the simplices ${}^i\sigma = \{{}^iv\colon v \in \sigma\}$, $\sigma \in K$, on the vertices ${}^iv = (i,v)$, $v \in \mathrm{vert}(K)$.

In case each $L_i = {}^i(K_i)$, the ith copy of a complex K_i, their join $*_{i\in\pi}(L_i)$ or $L_a * L_b * \ldots$ will be denoted by $\prod_{i\in\pi}(K_i)$ or $K_a \cdot K_b \ldots$ and its simplices $\bigcup_{1\le i\le p} {}^i(\sigma_i)$ will be denoted by $\prod_{i\in\pi}(\sigma_i)$ or $(\sigma_a, \sigma_b, \ldots)$, $\sigma_i \in K_i$. If all K_i's are equal to K, then this complex $\prod_{i\in\pi}(K_i) = K\cdot K\cdot \ldots$ will also sometimes be denoted by K^P.

Here $P = \overline{\pi}$, *the closed* (p-1)*-simplex* consisting of *all* subsets of π. More generally, given *any* simplicial complex $B \subseteq \overline{\pi}$, we define the **B-fold join** K^B **of** K to be the subcomplex of K^{π} consisting of all $\prod_{i\in\pi}(\sigma_i)$ such that $\{i\colon \sigma_i \neq \emptyset\} \in B$.

(2.4) Proposition. *Let* W *be a map, which associates to each simplicial complex* E *having vertices of the type* ${}^iv$, $1 \le i \le p$, *a stellar subdivision* W(E). *If* W *is*

(a) natural, i.e. $F \subseteq E$ implies $W(F) \subseteq W(E)$, *and*

(b) p-multiplicative, i.e. $(\bigcup_i(K_i)) \cap (\bigcup_i(L_i)) = \{\emptyset\}$[5] implies

$$W((K_1*L_1)\cdot \ldots \cdot(K_p*L_p)) = (W(K_1\cdot \ldots \cdot K_p)) * (W(L_1\cdot \ldots \cdot L_p)),$$

then W *is determined uniquely by its restriction to all closed* (p-1)*-simplices of the type* $v\cdot \ldots \cdot v$, *and these simplicial subdivisions* $W(v\cdot \ldots \cdot v)$ *can be arbitrary.*

Proof. Since any E is the union of its closed simplices $\overline{\sigma}_1\cdot \ldots \cdot\overline{\sigma}_p$, its subdivision W(E) will be determined if we can check that there is a unique, and natural, way of subdividing closed simplices.

To see this note that each factor $\overline{\sigma}_i$ of $\overline{\sigma}_1\cdot \ldots \cdot\overline{\sigma}_p$ is the join of (the 0-dimensional simplicial complexes determined by) its vertices v. So, by p-multiplicativity, $W(\overline{\sigma}_1\cdot \ldots \cdot\overline{\sigma}_p)$ can be written uniquely as the join of some W(F)'s, where each F is a face of some $v\cdot \ldots \cdot v$. But each

[5] Note that this is stronger than requiring that $K_1\cdot \ldots \cdot K_p$ and $L_1\cdot \ldots \cdot L_p$ be disjoint.

such W(F) is known since, by naturality, it has to be the restriction of the chosen subdivision of $v\cdot \ldots \cdot v$ to this face F.

Since a join of stellar subdivisions is a stellar subdivision of the join, it follows that $W(\bar\sigma_1\cdot \ldots \cdot\bar\sigma_p)$ is a stellar subdivision of $\bar\sigma_1\cdot \ldots \cdot\bar\sigma_p$. Also this procedure of subdividing closed simplices is clearly such that if $\bar\theta_1\cdot \ldots \cdot\bar\theta_p$ is a face of $\bar\sigma_1\cdot \ldots \cdot\bar\sigma_p$, then its subdivision $W(\bar\theta_1\cdot \ldots \cdot\bar\theta_p)$ is a subcomplex of $W(\bar\sigma_1\cdot \ldots \cdot\bar\sigma_p)$.

Thus W(E) is uniquely determined, as above, for *any* simplicial complex E. Also it follows easily, from our description of this subdivision process $E \longmapsto W(E)$, that it is p-multiplicative. *q.e.d.*

A function W of the above kind will be called a **generalized Wu subdivision of order** p if, for any two vertices u and v, $u \mapsto v$ induces an isomorphism of $W(u\cdot \ldots \cdot u)$ with $W(v\cdot \ldots \cdot v)$. Thus **generalized Wu subdivisions of order** p **are in one-one correspondence with subdivisions of the closed** (p-1)-**simplex** on the vertices {1, .. ,p}.

(2.5) Corollary. *For any simplicial complex* K, *the simplicial complex*

$$\{(\sigma_\pi, \sigma_1, \ldots ,\sigma_p): \sigma_1\cap \ldots \cap\sigma_p = \emptyset,\ \sigma_\pi\cup\sigma_i \in K,\ 1 \le i \le p\},$$

is a subdivision of its p*-fold join* $K\cdot \ldots \cdot K$.

This is the *joins version of* the **Wu subdivision** of the p-fold product $K \times \ldots \times K$ mentioned in §1.

Proof. For any E on vertices of type ${}^iv$, $1 \le i \le p$, consider the following complex on vertices of the type ${}^iv$, $1 \le i \le p$ or $i = \pi$.

$$W(E) = \{(\sigma_\pi, \sigma_1, \ldots ,\sigma_p): \sigma_1\cap \ldots \cap\sigma_p = \emptyset,\ (\sigma_\pi\cup\sigma_1, \ldots ,\sigma_\pi\cup\sigma_p) \in E\}.$$

We note that $E \mapsto W(E)$ is natural and p-multiplicative. Furthermore if $E = v\cdot \ldots \cdot v = {}^1v\cdot \ldots \cdot {}^pv$, then W(E) is same as the complex obtained by deriving it at the barycentre ${}^\pi v$ of the top-dimensional simplex.

So each W(E) is a stellar subdivision of E. The result follows because $W(K\cdot \ldots \cdot K)$ coincides with the given complex. *q.e.d.*

(2.6) Corollary. *Let* $\mathcal{P}$ *be the poset of all nonempty subsets of* π = {1, .. ,p} *under* $\subseteq$. *Then, for any simplicial complex* K, *the simplicial complex consisting of all simplices* $\prod_{\alpha\in\mathcal{P}}(\sigma_\alpha)$ *where* $\sigma_\alpha\cap\sigma_\beta = \emptyset$ *whenever* α *and* β *are incomparable and* $\cup_{\alpha\in\mathcal{C}}(\sigma_\alpha) \in K$ *for all chains* $\mathcal{C} \subseteq \mathcal{P}$, *is a stellar subdivision of the p-fold join* $K\cdot \ldots \cdot K$.

This subdivision is due to **T.Bier** who was kind enough to tell me about it while we were discussing these questions in 1992.

Proof.

(2.6) Likewise let $\mathcal{P}_j$ be poset of subsets of π of cardinality $\geq$ j or singletons. A similar description can be given ?

(2.7) We now turn our attention to j-fold join of a space X

(2.8) We will now establish the generalized Wu lemma stated in the introduction.

§3. Remarks

References

S.-T. Hu, J.F.P. Hudson (replace this by original source), M.H.A. Newman, A. Shapiro, W.-T. Wu, —,