aft FTA

In note ढ we'd worked out *the topology of the space of all unordered n-tuples of points from euclidean m-space*. Surprisingly I did *not* find this result in the literature, but I've found much else – and remembered even more – during this detailed search, so I'll continue that miscellany of thoughts in the same vein for some time more before I wrap up this extension of note घ, to return again—see ५—to the cartesian genesis and evolution [1] of closed manifolds.

११) Symmetric squares of spheres occur as his *huitième exemple* in §15 of Poincaré's *Analysis Situs* (1895). A very thorough discussion of this and his preceding example—the tight embedding $\pm(x,y,z) \mapsto (x^2,y^2,z^2,xy,yz,zx)$ of $\mathbb{R}P^2$ in the 5-dimensional linear subspace on which the first three coordinates sum to one—together with extensive remarks on later developments, can be found on pages 94 to 102 of my seminar notes, *Poincaré's Papers on Topology* (1993-94); and also on pages 136, 137 and 146 of an abbreviated later write-up, *The topological work of Henri Poincaré* (1999). The reader is strongly urged to at least browse these cited pages before continuing further.

१२) The funny thing is I myself had *completely forgotten* the above when I was working out—doggedly determined to look up the literature only *after* I had worked them all out—the proofs given in the last installment. I could not even recall if I had ever seen explicitly mentioned—I sure had, it is on the aforementioned page 146 of my *own* paper—that the symmetric powers of any 2-manifold are manifolds. Of course, the moment I put this question later to the web, I was reassured, but the above scanned items on my website did not show in this search : it was two weeks more before it occured to me that Poincaré must have said something about symmetric powers.

१३) It was known to Pontryagin in the 1930's that, *the space of all conjugate pairs of quadratic equations over the complex numbers is a 4-sphere*. Indeed, $(z_1,z_2,z_3) \mapsto (z_1\overline{z}_1, z_2\overline{z}_2, z_3\overline{z}_3, \frac{1}{2}(z_1\overline{z}_2+\overline{z}_1 z_2), \frac{1}{2}(z_2\overline{z}_3+\overline{z}_2 z_3), \frac{1}{2}(z_3\overline{z}_1+\overline{z}_3 z_1))$ maps—Kuiper (1973)—the unit sphere of $\mathbb{C}^3$ onto the boundary of the convex hull of Poincaré's embedded $\mathbb{R}P^2$. Alternatively, it is not hard to check that, *the symmetric nth power of $\mathbb{R}P^2$ is $\mathbb{R}P^{2n}$*, and $\mathbb{C}P^2/\text{conjugation} \cong S^4$ can be deduced as a corollary—Massey (1972)—of the case $n = 2$ of this result.

१४) In the seminar notes $\mathrm{Sym}^n(X)$ denoted *the Polish nth symmetric power*, i.e., all subsets of cardinality $\leq n$ of X, we'll now denote it by $\mathrm{Sym}^n_P(X)$ and use $\mathrm{Sym}^n(X) := X^n/\Sigma^n$ for all unordered n-tuples of X. For $n > 2$ the two are different, for example the natural surjection $\mathrm{Sym}^3(X) \to \mathrm{Sym}^3_P(X)$ maps the two unordered 3-tuples $\{a,a,b\}$ and $\{a,b,b\}$ to the same cardinality 2 subset $\{a,b\}$. Indeed, *the space $\mathrm{Sym}^3(S^1)$ of cubic equations over the reals with all roots real is a solid torus*, on its boundary 2-torus a root is repeated and $\{a,a,b\} \leftrightarrow \{a,b,b\}$ switches latitude and longitude; by identifying these (as Keerti had last year checked) one gets $\mathrm{Sym}^3_P(S^1) \cong S^3$, *as was shown by* Bott (1953) *in a letter to* Borsuk *who four years before had claimed it was $S^1 \times S^2$.*

[1] As developed in various notes of *Plain Geometry & Relativity, I-V*.

੧੫) The homeomorphism $\mathrm{Sym}^n(\mathbb{R}P^2) \cong \mathbb{R}P^{2n}$ of ੧੩ is *not* multiplication of n real quadratic (homogenous in x and y) equations—this factorization is not unique—but there is this *fundamental partition of* $\mathbb{R}P^n$: subspace $\mathrm{Sym}^n(S^1)$ of real degree n equations with all roots real, then those with one pair of conjugate complex roots, then those with two, etc. The lifted centrally symmetric partition of S^n for $n = 2$ is shown in ਮੂਲ ਮਸਲਾ; and for $n = 3$ all real cubic equations with real roots lift to a solid torus in S^3 :- the remaining equations factorizing into that yellow open 2-cell B^2 worth of quadratic equations with complex roots, times an S^1 of linear equations, i.e., an open solid torus $\square$; but what then is the topology of this natural partition of the n-sphere for $n \geq 4$?

੧੬) We know–note ਙ–that $\mathrm{Sym}^n(\mathbb{R}^m)$ *is the infinite cone over the join* $J^{n,m}$ *of* S^{m-1} *and* $n-1$ *copies of* $\mathbb{R}P^{m-1}$. Using $S^m = \mathbb{R}^m \cup \{\infty\}$ it follows that, $\mathrm{Sym}^n(S^m), n \geq 2$, *is the mapping cone of a surjective 'hopf map'* $h^{n,m} : J^{n,m} \to \mathrm{Sym}^{n-1}(S^m)$, generalizing case $m = 2$, when $J^{n,2} = S^{2n-1}$, $\mathrm{Sym}^n(S^2) = \mathbb{C}P^n$ and $h^{n,2} : S^{2n-1} \to \mathbb{C}P^{n-1}$ is the usual fibration into circles.

੧੭) Despite many instances like ੧੨ above, I'm often told I have a very good memory (!) but neither perhaps should I call my memory 'bad' : forgetting has helped me discover some new mathematics by seeing things a bit differently, for example, *seeing projective spaces as 'spaces of equations'* as in note ਦ is pretty useful. Indeed, as I have suggested before—see notably ਟ, ਠ and ਡ—our vaunted mental progress is more a 'recycling in time'.

੧੮) Poincaré complained of a bad memory, likewise Atiyah [2], yet at 88 (!) he is still making beautiful mathematics, e.g., a simply connected closed manifold $\mathbb{M}^4(\mathit{Helium})$ with $e = 8, \tau = 0$, which is a 2-fold branched cover of the 4-torus, and has the quaternion group of order 8 acting freely on it, and which, per a numerology—see ੫—that he has been finessing since 2011, tells what the second element looks like. For the first, $\mathbb{M}^4(\mathit{Hydrogen}) = \mathrm{Sym}^2(S^2) = \mathbb{C}P^2$, and it seems he is mulling for the other elements, simply connected closed manifolds branch covering the squares $\mathrm{Sym}^2(M^2)$ of surfaces of genus ≥ 2 ?

੧੯) Now there are more than four, etc., but Atiyah's elements are akin to those in Euclid's *Elements*.[3] In *Geometrical microphysics* (1975-77) I too had interpreted the kinematics and dynamics of a microsystem as the topology and geometry of a closed manifold which cuts physical space (spacetime?) in an M^2 shaped microscope, with only those cohomology classes of an ergodic microflow observable which can be defined in terms of a triangulation of the manifold. In writing this mouthful I've refreshed my memory by reading and looking at a

[2] Atiyah once said that, but for his bad memory, he could well have become a historian, but methinks—and my two papers entitled *"The Forgotten ... "* should clinch this point!—that a bad memory is no impediment to doing good history either.

[3] For more on this 'recycling in time' see pages 7 and 11 of *"213, 16A" and Mathematics* (2010). This watershed paper contains a lot, e.g., a construction of not only the five platonic solids but of many, many more—see also *How I learnt some well-known folklore* (2010)—in its mere 37 pages. Despite appearances, this and most subsequent papers on my website are *not* typical postings that you can get a hang of by browsing on-line. If you want to understand what I have been doing lately, you'll need to print out these pdfs (in colour!) and peruse them : a continuous but circular story with flashbacks is being narrated in a *condensed* style, evenso, a beautiful and simple yet broad picture of it all is emerging ...

picture in an extract which appeared years later as *Stationary states* (2002); the rest I've still to dig up, organize and post on my website; but yes, the energy levels of the hydrogen atom were tied to the homotopy classes of maps from a hidden 3-sphere to a 2-spherical microscope. Since $\mathbb{C}P^2$ is the mapping cone of a generating map $S^3 \to S^2$ seems I was on the right track, but, if Atiyah's hydrogen cuts physical spacetime tightly in an $\mathbb{R}P^2_6$, may be we should hone it to the $\mathbb{C}P^2_9$ of Kühnel and Banchoff (1983) obtained by splitting three of the six antipodal pairs of vertices of the icosahedron, i.e., Euclid's water [4] ...

૨૦) ... with perhaps the two unequal orbits of the action of the heisenberg group over $\mathbb{Z}_3$ on $\mathbb{C}P^2_9$ the proton and electron of this hydrogen? Heisenberg's microphysics was linear but infinite dimensional. Schrödinger's had, as well, a finite dimensional but nonlinear dual. Which was declared not worth bothering about. So, naturally, all but the very obedient started bothering about it. My microphysics was on smooth closed manifolds of any finite dimension, with a modicum of geometry. Now, starting from even less—e.g., see *PG&R Note 7*—we have understood broadly, the cartesian creation and evolution of *all* closed manifolds, see *PG&R-V Note 28*. Atiyah's atoms, on the other hand, are closed smooth manifolds of dimension $2+2 = 2\times 2$, and carry the yang-mills geometry which uses the self-duality available only in this dimension. This had enabled Donaldson (1983) to show that a closed simply connected smooth 4-manifold is *orientably* cobordant to some $\mathbb{C}P^2$'s iff its intersection form is positive definite, thus giving us some but not, *all* the 4-dimensional simply connected manifold matter that can evolve smoothly from Atiyah's hydrogen? [5]

૨૧) [6] Continuing ૧૫, the subspace of all real degree n equations with i pairs of conjugate complex roots is homeomorphic to $\mathrm{Sym}^{n-2i}(S^1)\times B^{2i}$:- they factorize uniquely into $n-2i$ linear equations, which gives $\mathrm{Sym}^{n-2i}(S^1)$, and i quadratic equations with complex roots, which gives $\mathrm{Sym}^i(B^2) \cong B^{2i}$ $\square$; and the union of these $[\frac{n}{2}]+1$ disjoint subspaces is $\mathbb{R}P^n$, the space of all real degree n (homogenous in x and y) equations. So, for the topology of these parts it suffices to know the symmetric powers of S^1, but for how they fit, we'll need to look also at how many conjugate pairs of roots approach a repeated real root, etc. We know that $\mathrm{Sym}^n(S^1)$ is an n-dimensional manifold-with-boundary :- for $\mathrm{Sym}^n(\mathbb{R}), n \geq 2$, is a closed half n-space by note ૫ $\square$; its interior is all degree n equations with n *distinct* real roots. Also for $n = 2$ and 3 we know that $\mathrm{Sym}^n(S^1)$ is, respectively, a möbius strip and a solid torus. Indeed, for any n, $\mathrm{Sym}^n(S^1)$ is homeomorphic to the mapping torus of the antipodal map of the closed ball D^{n-1}. To see this we'll use $S^1 \cong \mathbb{R}/\mathbb{Z}$—as was done for $n = 2$ on page 99 of the Poincaré seminar notes—and examine the defining identifications of $\mathrm{Sym}^n(\mathbb{R}/\mathbb{Z})$:-

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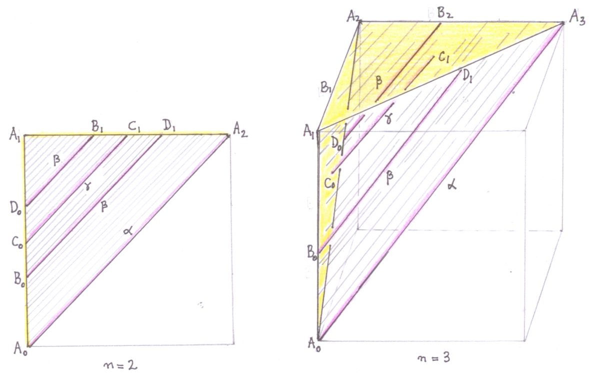
[4] The four greek elements were the four humours of unani medicine, which is still popular, and icosahedral calthrates of water have been observed by fans of homoeopathy.

[5] *The de Rham cohomology of foliated manifolds* (1974) is a Stony Brook thesis, but I became aware of the excitement only later when someone there, on receiving *Geometrical microphysics*, had advised me to *jump on this yang-mills bandwagon!* However Thom had empathized with my cartesian approach, but by the time I received his insightful suggestions—the use of 'dual' in this paragraph was based on something he wrote—it was a wee bit too late.

[6] On we go! but it is clear by now that footnote 20 of *PG&R-V* was far more than an aside, and, we have returned to the genesis and evolution of closed manifolds.



२२) Each equivalence class $\pi(x_1, \ldots, x_n)$, where x_i are real numbers mod 1, has one member such that $0 \le x_1 \le \cdots \le x_n \le 1$, and only one if in the interior of the n-cube, $0 \le x_i \le 1$, but has more if on its boundary, so identifications need to be made here. To wit, if A_i denotes the vertex with last i coordinates 1, $\mathrm{Sym}^n(S^1)$ is the quotient space of the closed n-simplex $A_0 \ldots A_n$ obtained by identifying a pair of ordered facets : $A_0 \ldots A_{n-1} \equiv A_1 \ldots A_n$. This 'increasing' n-simplex is one of the $n!$ into which the n-cube is subdivided by the possible orders of the n components. We'll now partition it into segments parallel to the edge A_0A_n to which its remaining $n-1$ facets are incident. The ends of the segment C_0C_1 joining the centres of $A_0 \ldots A_{n-1}$ and $A_1 \ldots A_n$ get identified, and any other segment gets concatenated with some others before it closes with total length a bigger mutiple of C_0C_1 dividing A_0A_n. This partition into circles shows $\mathrm{Sym}^n(S^1)$ as the mapping torus of the homeomorphism of the closed $(n-1)$-ball $A_0 \ldots A_{n-1}$ linearly extending the permutation $A_1 \ldots A_{n-1}A_0$ of vertices. This permutation is even, so the homeomorphism orientation preserving, iff n is odd; hence, as far as the topology of the space is concerned, we can here use instead the antipodal homeomorphism. □

२३) So, for n odd, $\mathrm{Sym}^n(S^1) \cong S^1 \times D^{n-1}$, e.g., identifying $ABC \equiv BCD$ a tetrahedron $ABCD$ becomes a solid torus; and—see १४—if also $CDA \equiv BDA$, then an S^3; on the other hand, the identifications $ABC \equiv BCD, CDA \equiv DAB$ give an orientable 3-manifold with fundamental group $\mathbb{Z}_4$:- These identifications, $I \equiv I'$ and $II \equiv II'$, of the four facets of the tetrahedron, partition its six edges into two cycles, $II' \cap I \equiv I' \cap I \equiv I' \cap II \equiv II' \cap II$ and $I \cap II \equiv II' \cap I'$, so the dual generators I and II of this group are subject only to the two relations, $I + I + II + II = 0$ and $II - I = 0$, i.e., $II = I$ and $4I = 0$. The quotient space has no singularities and is orientable because [7] $e = 1 - 2 + 2 - 1 = 0$ and $\{I, I'\}$, $\{II, II'\}$ have opposite signs in $\partial(ABCD)$. □ If $ABC \equiv BCD, CDA \equiv BDA$, the edges partition into $II' \cap I \equiv I' \cap I \equiv I' \cap II \equiv II' \cap I' \equiv I \cap II$ and $II' \cap II$, which give $I + I + II - I + II = 0$ and $II = 0$, so $I = 0$. □ Besides we can make two non-orientable 3-manifolds from $ABCD$, also covered by S^3; and maybe,

[7] See Combinatorial methods in topology (1994), page 60; and on its page 62 are, *the seven orientable 3-manifolds that can be made by identifying the opposite facets of a cube*.

all odd dimensional closed pseudomanifolds made by identifying pairs of facets of a simplex have fundamental group finite?

૨૪) For example, identifying facets $A_0 \dots A_{n-1} \equiv A_1 \dots A_n$ of the n-simplex, $A_0 \dots A_n$, then $A_2 \dots A_n A_0 \equiv A_3 \dots A_n A_0 A_1$, etc., gives a pseudomanifold with fundamental group $\mathbb{Z}_{(n+1)/2}$ for any odd $n > 3$; but now there are singularities, indeed $e \neq 0$; and maybe, only finitely many (closed) manifolds can be made over all n, by identifying facets, from the n-simplex? However, any cartesian power of our closed string S^1, symmetrized only with respect to suitable permutation subgroups and their spin covers, should give cartesian manifold matter like Atiyah's helium—see ૧૮—which can be obtained from the 4-torus by using the spin cover of the alternating group on four letters. Also, it should be possible to make these n-manifolds from the n-cube, just like how S^3 divided by the quaternion group of order eight—see the Poincaré seminar notes—was made by an identification of the opposite facets of the 3-cube.

૨૫) So, continuing ૨૧, each part $\tilde{P}_i$ of the lifted fundamental partition of S^n is topologically the same open manifold $S^1 \times B^{n-1}$ plus varying amounts of its boundary, except, for n even, the final part is two open n-balls, $S^0 \times B^{n-1}$, while for n odd it is $S^1 \times B^{n-1}$. The initial part P_0 of all degree n real equations with no complex roots we know is $\mathrm{Sym}^n(S^1)$, a manifold-with-boundary; so we ask, is the union of the first some parts also one; i.e., is the subspace $W_i \subseteq \mathbb{R}P^n$ of all degree n real equations with at most i conjugate pairs of complex roots a manifold-with-boundary? The answer seems to be yes :- We assume inductively that ∂W_{i-1} is a closed $(n-1)$-manifold. The closed subset W_i of $\mathbb{R}P^n$ is the union of W_{i-1} and $P_i \cong \mathrm{Sym}^{n-2i}(S^i) \times B^{2i}$. Some of the boundary points of $\mathrm{int}P_i$ are in P_i, these form an open $(n-1)$-manifold $\partial(\mathrm{Sym}^{n-2i}(S^i)) \times B^{2i}$. The remaining boundary points of P_i in $\mathbb{R}P^n$ form ∂W_{i-1}. Of these, an open $(n-1)$-manifold worth of points is now in the interior of W_i, viz., those corresponding to degree n equations with $i-1$ complex conjugate pairs of roots and at most three equal real roots: of these two bifurcate to the ith pair of complex roots. The remaining boundary points x of P_i complete ∂W_{i-1}. The topology of ∂W_i near this equation x is like that of ∂W_{i-1} near x, for, if all equations x' with same roots as x except (r, r) which are now (r_1, r_2) at an $\epsilon > 0$ formed a circle of that spherical link, all equations x'' with these roots now $(z, \overline{z})$ at this distance form a circle; so ∂W_i is also a closed $(n-1)$-manifold. □

૨૬) So, can *any* closed n-manifold – the child of a periodic cartesian flow in a higher dimensional spacetime – be decomposed similarly into these *curled-up spacetimes* $S^1 \times B^{n-1}$—*PG&R-IV, Figure 9, Notes 25-27*—if $e = 0$, and if not also some open balls? Then, within these curled-up spacetimes is born, from their chaotic cartesian flows, still smaller cartesian matter ... Adding P_i is not quite the same as attaching a round handle of index $2i$ to W_{i-1}, but as far as the topology of W_i is concerned, it is : for n odd the fundamental partition gives an economical round handle decomposition of $\mathbb{R}P^n$ and S^n. That round handle decompositions exist under the obvious necessary conditions was Asimov, and then Thurston had made, using his mysterious local construction, a codimension one foliation from any round handle decomposition. Maybe for the fundamental partition there is no mystery, indeed this spherical partition seems tied to how, Lawson and Tamura had previously foliated all odd spheres, and maybe it will give us even nicer, foliations of odd spheres tied to the discriminants which occur in the sturm theory of polynomial equations?

੨੭) Equations of degree at most four were solved by Khayyam[8], and I think this aesthete would have loved our elaboration, this fundamental partition of the n-spheres, the simplest of all cartesian matter, into essentially just $(n+1)/2$ round spacetimes, in which our doomed urge to hold on to the here and now, absolute time = constant, is itself represented by, reeb snakes trying to swallow their own tails, an apt motif for the constant recycling of time in our minds, as much as in these tiny parts of matter.

੨੮) *Two questions*—a foliation on S^3 ? a complex structure on S^6 ?—were raised by Hopf in the 1930's. Foliated and complex structures are alike, they can both be viewed as Lie algebras of vector fields that a manifold can admit only if it satisfies some obvious conditions: these S^3 and S^6 respectively fulfil, but, *for S^6 the answer is still blowing in the wind !* For more on this viewpoint see *Non-degenerescence of some spectral sequences* (1984), the notion of a complex structure allows numerous examples [9] very different from the nice ones that arise in algebraic geometry. These nice ones S^6 can't admit, and the constructions of possible complex structures on it that have been proposed are very intricate indeed; on the other hand, Atiyah (2016) has proferred a mod 2 invariant which will prove that S^6 does not admit *any* complex structure !

੨੯) The fundamental partition depends on a total order : a point of $\mathbb{R}P^n$ was deemed an equation of degree n by making its coordinates in order the $n+1$ coefficients. Which echoes many things in that omnibus paper, *On neighbourly triangulations* (1983), for example, order-orientable triangulations and heawood inequalities, the latter because some algebraic shifting, order-dependent yet such that it does not kill van Kampen's mod 2 invariant, will do the job (note a total order has given us now a $\mathbb{Z}/2$-partition of the sphere). Also, Atiyah's strategy for S^6 echoes non-embeddability in twice dimensional space using van Kampen's invariant. One checks for one immersion it is nonzero, and then uses the fact that the immersion is unimportant. Likewise Atiyah checks for one almost complex structure on S^6 that his definition does give an odd number, and then claims that the almost complex structure is unimportant. His definition is of course different – the key seems to be his profound paper *K-theory and reality* (1966) (which seems related to the fundamental partition) – and ways more subtle. Which reminds me that there have been around now for many decades some other subtle mod 2 invariants, notably that of Kirby and Siebenmann, which too I have never quite understood to my satisfaction, so there is much left to do, and my plate remains as full as ever ...

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[8] See Amir-Moéz (1962), also ਓਮਾਰ ਖਿਆਮ ਦੀ ਸੱਭ ਤੋਂ ਪਿਆਰੀ ਰੱਚਣਾ.

[9] For similar compact examples, I was planning to use the complex structures defined by Wang on symmetric spaces, but this sequel was not written up.