FROM CALCULUS TO CYCLIC COHOMOLOGY

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by

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Introduction. The main object of this talk is to point out how homology theory arose, in a perfectly natural way, from the calculus, and to trace the main thread of ideas which led, about a hundred years later, to the consideration of its cyclic variants.

Also, motivated by a formula which occurs in §7 of Poincaré's *Analysis Situs*, 1895, I'll define in the end some new cohomologies. This will show clearly that *these cyclic variants could have been discovered much earlier* if a little more attention had been given to the way in which Poincaré had written this formula.

§1. **MULTIPLE INTEGRALS.** Our story begins in 1895 with a definition of POINCARÉ (§7 of his *Analysis Situs*) which I'll rephrase in the following intuitively suggestive way.

DEFINITION. *The rth Betti number of M is the ambiguity of a locally unambiguous generic r-fold indefinite integral I on M.*

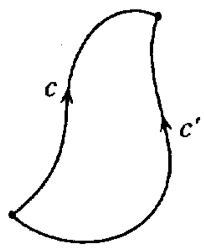
EXPLANATION. Here M is a smooth compact manifold which comes embedded in an $\mathbf{R}^N$ and the **indefinite integral** is given by

$$I = \int \cdots \int \sum X_{\alpha_1 \cdots \alpha_r} dx_{\alpha_1} \cdots dx_{\alpha_r},$$

where the functions $X_{\alpha_1 \cdots \alpha_r}$, $1 \leq \alpha_i \leq N$, are assumed defined and smooth on a "very small" (= tubular) open neighbourhood U of M in $\mathbf{R}^N$.

The adjective "indefinite" of course signifies that no **domain of integration** c was specified for I. Regarding these we will make the *simplifying assumption* that M comes equipped with a cell subdivision K, and that the permissible c's on which I is to be evaluated are linear combinations of oriented r-cells of K.

Now you are doubtless familiar with the fact that a line integral (case $r = 1$), when evaluated over two paths c and c' having the same initial and the same final point, can give different answers.



A **locally unambiguous** integral will be one for which the above *cannot happen locally*, i.e. its value on any two "small" domains of integration c and c' is the same whenenver $\partial c = \partial c'$.

We recall next (cf. any "advanced calculus" book) that, for the cases $N = 3$, $r = 1$, and $N = 3$, $r = 2$ respectively, i.e. for line and surface integrals of 3-space, one has the following.

$$\int P\,dx + Q\,dy + R\,dz \quad is\ locally\ unambiguous\ iff$$

$$\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} = \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} = \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 0.$$

$$\int\int P\,dydz + Q\,dzdx + R\,dxdy \ is\ locally\ unambiguous\ iff$$

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0.$$

Poincaré criterion (*Acta Math.* 1882) says more generally that the above r-fold indefinite integral I is locally uambiguous iff

$$\frac{\partial(\alpha_1, \cdots, \alpha_r)}{\partial[\alpha_{r+1}]} \pm \frac{\partial(\alpha_2, \cdots, \alpha_{r+1})}{\partial[\alpha_1]} \pm \cdots = 0. \qquad (*)$$

The above formula is written almost exactly as given in the ***Analysis Situs:*** $(\alpha_1, \cdots, \alpha_2)$ is Poincarés abbreviation for $X_{\alpha_1 \cdots \alpha_r}$ and $[\alpha_i]$ for x_{α_i}, and the signs in the above *cyclic sum* are all positive if r is even, and alternately positive and negative if r is odd.

Now choose any maximal set of integrally independent "closed domains" ($\partial c = 0$) of integration c_1, $c_2, \cdots$ (note that this set is finite because of our simplifying assumption) and let p_1, $p_2, \cdots$ be the values of I on these. Using what went into Poincaré proof of $(*)$ (for a mordernized sketch of this see §2) it follows that the values of I over *any* c and c' having $\partial c = \partial c'$ can differ by at most an integral linear combination of these **periods** p_i.

Thus the rank $b_r(I)$ of the additive subgroup of $\mathbb{R}$ generated by the p_i's– incidentally this is called the *period subgroup* of I – is a good measure of the (global) **ambiguity** of our I. A **generic** I is one with $b_r(I)$ maximal, and then we have set $b_r(I) = b_r(M)$. This finishes our explanation of the informal definition given above (see also Historical Remarks given in the end).

§2 **DIFFERENTIAL FORMS.** The quickest way to "define" these is to say simply that they are obtained from indefinite integrals by *"erasing the integral signs"*, so e.g. I gives

$$\omega = \sum X_{\alpha_1 \ldots \alpha_r} dx_{\alpha_1} \wedge \cdots \wedge dx_{\alpha_r}.$$

In other words forms are assumed endowed with the algebraical properties dictated by the properties of integrals. Thus the **change of variables formula** of integration tells us how they must transform under a change of coordinates, in particular $X_{\alpha_1 \cdots \alpha_r}$ must be *totally skewsymmetric* in the indices, one must have $dx \wedge dy = -dy \wedge dx$, etc.

To understand Poincaré's condition $(*)$ CARTAN defined the **exterior derivative** of forms $\Omega^* \xrightarrow{d} \Omega^{*+1}$ by

$$d\omega = \sum d\left(X_{\alpha_1 \cdots \alpha_r}\right) \wedge dx_{\alpha_1} \cdots \wedge dx_{\alpha_r}.$$

Here $\omega \in \Omega^r$ (i.e. ω is any r-form on U) and $d(f)$ denotes the *total differential* of the function f.

An elementary calculation now shows that the left side of $(*)$ is precisely the $(\alpha_1, \cdots, \alpha_{r+1})$th coefficient of $d\omega$. *Thus in Cartan's notation Poincaré's* $(*)$ *becomes simply* $d\omega = 0$.

We can now quickly sketch (in Cartan's abbreviated notation) *Poincaré's proof of* $(*)$. First, generalizing (by using) the **fundamental theorem of calculus,** he got

$$\int_{\partial c} \omega = \int_c d\omega$$

(here $\int_c \omega$ denotes the value of the corresponding I on c). From this the necessity of $(*)$ is immediate. Next he showed that $(*)$ implied the *local solvability* of the differential equation $d\theta = \omega$ (this implication is now called **Poincaré Lemma**) and then again used the above formula to obtain the sufficiency of $(*)$. *q.e.d.*

We turn next to *Cartan's reformulation of Poincaré's definition of* $b_r(M)$. For this note that, but for genericity, the $I's$ which were used in the definition of §1 are precisely those which constitute $\ker(d)$. So evaluating each $\omega \in \ker(d)$ on the basic closed domains of integration $c_1, c_2, \cdots$ we can define a surjection of $\ker(d)$ onto a vector space of dimension $b_r(M)$ whose kernel contains $\mathrm{im}(d)$. In fact (this requires care) this kernel *equals* $\mathrm{im}(d)$: so the *Betti numbers* $b_*(M)$ *coincide with the graded dimensions of the* **De Rham cohomology** $H^*_{DR} = \frac{\ker(d)}{\mathrm{im}(d)}$ of M (the name being after the mathematician who "took care" of the above point).

§3. **CURRENTS** are simply things *dual* to forms i.e. they are linear functionals $C : \Omega^* \to \mathbb{C}$. However some care is necessary here, because Ω^* being *not* finite-dimensional, using *all* linear functionals *will not* result in duality. Following SCHWARTZ a way out is to demand that each C be also *continuous* with respect to natural C^∞ *topology* on Ω^*. This saves the day because now the Hahn-Banach Theorem (which holds for any Hausdorff locally convex topological vector space) guarantees the requisite duality.

We will denote the dual complex of currents by $\Omega_* \xrightarrow{B} \Omega_{*-1}$; furthermore, $\frac{\ker(B)}{\mathrm{im}(B)}$ will be denoted $H^{DR}_*(M)$ and called the **De Rham homology** of M. Note that by definition $BC(\omega) = C(d\omega)$, so e.g. a "closed current" $(BC = 0)$ is one which vanishes on $\mathrm{im}(d)$.

Examples of currents. (1) Each r-dimensional domain of integration c can be identified with the r-dimensional current $\theta \mapsto \int_c \theta$. Under this identification ∂c becomes Bc. (2) Also, using the Euclidean metric on forms of $\mathbb{R}^N$ and the Lebesgue measure of $\mathbb{R}^N$, one can identify r-forms ω with the corresponding currents $\theta \mapsto \int \langle \omega, \theta \rangle \, d\mu$. We note that this gives a dense subspace of Ω_*. Thus one can alternatively also think of the passage from forms to currents as a *completion* process. (We note thus that, by going to currents, Schwartz sort of re-inserts the integrals which Cartan had erased !)

§4. **CHARACTERS.** We now come to CONNES who was interested in some *"quantum spaces"* (e.g. the "space of leaves" of a foliation). Then one has no (reasonable) M, but there is a (non-commutative) analogue of the algebra $\mathcal{A} = C^\infty(M)$! So he wanted to reformulate the above definition of homology entirely in terms of $\mathcal{A}$ in a way that would make sense even for non-commutative algebras.

To do this he replaced — I am here following Connes' book *Géométrie Non Commutative* — each r-dimensional current C by the degree $r+1$ multilinear forms $\tau : \mathcal{A} \times \cdots \times \mathcal{A} \to \mathbb{C}$ given by

$$\tau\left(f_0, f_1, \cdots, f_r\right) = C\left(f_0 df_1 \wedge \cdots \wedge df_r\right).$$

Proposition 1. *If τ arises from an r-dimensional current C as above then it must be a* **character** *of $\mathcal{A}$, i.e. we must have*

$$\tau(f_0 f_1, f_2, \cdots, f_{r+1}) - \tau(f_0, f_1 f_2, f_3, \cdots, f_{r+1}) + \cdots$$

$$\pm\tau(f_0, f_1, \cdots, f_{r-1}, f_r f_{r+1}) \mp \tau(f_{r+1} f_0, f_1, \cdots, f_r) = 0, \qquad (**)$$

for all $f_0, f_1, \cdots, f_{r+1} \in \mathcal{A}$.

Proof. For $r = 1$ the condition is essentially $\tau(1f, g) - \tau(1, fg) + \tau(g1, f) = \emptyset$ and so is equivalent to the *product rule* $fdg - d(fg) + gdf = \emptyset$. Likewise for $r \geq 2$ the condition is true because all it is saying is that such a τ has to be zero on any "length $r+1$ relation" generated by the product rule : more explicitly if we use $d(f_i f_{i+1}) = f_i df_{i+1} + f_{i+1} df_i$ on all terms of the left side excluding the first and the last, then in the new expression each term will cancel with the next. *q.e.d.*

We remark that τ's arising from C's also satisfy *some other necessary conditions.* They obey a continuity condition parallel to the one imposed on the C's. They are normalized, i.e. $\tau(f_0, f_1, \cdots, f_r) = 0$ if, for some $i \geq 1$, f_i is a constant function. Further, they are obviously totally skewsymmetric in variables other than f_0.

On the other hand there is no reason that such a τ be skewsymmetric in *all* its variables, however something less is true in one important case.

Proposition 2. *A τ arising from a C is rotationally skewsymmetric iff C is closed.*

Proof. The required $\tau(f_0, f_1, \cdots, f_r) = (-1)^r \tau(f_r, f_0, f_1, \cdots, f_{r-1})$ is a consequence of the fact that $f_0 df_1 \wedge \cdots \wedge df_r - (-1)^r f_r df_0 \wedge df_1 \wedge \cdots \wedge df_{r-1}$ is $\pm$ of the exterior derivative of $(f_0 f_r) df_1 \wedge \cdots \wedge df_{r-1}$, and so the closed current C evaluates to zero on it. The converse is also clear. *q.e.d.*

Just like $(*)$ immediately led to the exterior derivative, $(**)$ now suggests the **Hochschild coboundary** $b : C^*(\mathcal{A}, \mathcal{A}^*) \to C^{*+1}(\mathcal{A}, \mathcal{A}^*)$. More precisely, we replace $\tau : \mathcal{A} \times \cdots \times \mathcal{A}$ ($r+1$ times) $\to \mathbb{C}$ by $T : \mathcal{A} \times \cdots \times \mathcal{A}$ (r times) $\to \mathcal{A}^*$, where

$$T(f_1, \cdots, f_r)(f_0) = \tau(f_0, f_1, \cdots, f_r),$$

and rewrite $(**)$ as $bT = 0$, where b is defined by

$$(bT)(f_0, f_1, \cdots, f_r) = f_0 T(f_1, \cdots, f_r) - T(f_0, f_1 f_2, \cdots) + \cdots$$

$$\pm T(f_0, \cdots, f_{r-2} f_{r-1}, f_r) \mp T(f_0, \cdots, f_{r-1}) f_r.$$

Here, in the first and the last terms, we have used the obvious left and right $\mathcal{A}$-action on the vector space $\mathcal{A}^*$ of all functional $\mathcal{A} \to \mathbb{C}$ (so all this makes sense even for non-commutative algebras).

One has $b \circ b = 0$ and so the **Hochschild cohomology** $\frac{\ker b}{\operatorname{im} b}$ is defined. Its importance for us stems from the following "Hodge theorem": *each Hochschild cohomology class contains one and only one T arising from some current.* So currents (resp. closed currents) can be replaced by (resp. rotationally skewsymmetric) Hochschild cohomology classes. Prompted by this it was natural for Connes to check and confirm the following striking fact.

The rotationally skewsymmetric τ's constitute a sub cochain complex of the Hochschild complex $(C^*(\mathcal{A}, \mathcal{A}^*), b)$. (We note in this context – cf §5 – that the totally skewsymmetric τ's *do not* in general constitute a sub cochain complex.) The cohomology of this sub cochain complex is called the **cyclic cohomology** $HC^*(\mathcal{A})$ of $\mathcal{A}$.

As one would expect from the above, $HC^*(\mathcal{A})$ turns out to be closely related to $H_*^{DR}(M)$: it is itself "bigger" but some "extra modding-out" (whose details we'll omit) gives the latter. Also it is known now how one can "lift" the current boundary B all the way to $C^*(\mathcal{A}, \mathcal{A}^*)$ to get a useful (B, b) *double complex* etc., etc.

However in this lecture I will go no further into these nuts-and-bolts of cyclic homological algebra – for this see e.g. some subsequent lectures of this Workshop and Loday's book, *Cyclic Homology* – but will instead return once again to the formula $(*)$ of §1.

§5. POINCARÉ COHOMOLOGIES. The interpretation of Poincaré's $(*)$ as $d\omega = 0$ in §2 leaves something to be desired : it does not fully "explain" the beautiful cyclic symmetry of its left side. We will now give another (and more straightforward !) interpretation of $(*)$ which does this, and leads to some interesting new cohomologies.

We consider the vector space $\Omega^r_{\text{assoc}}(U)$ of all functions X from all length r *index sequences* $\alpha_1\alpha_2\cdots\alpha_r$, $1 \le \alpha_i \le N$, to $\mathcal{A} = C^\infty(U)$. Of course if we assume skewsymmetry, $X(\alpha_{\pi1}\alpha_{\pi2}\cdots\alpha_{\pi r}) = (-1)^\pi X(\alpha_1\alpha_2\cdots\alpha_r)$, with respect to *all* permutations π of the sequences, the resulting subspace $\Omega^r_{\text{alt}}(U)$ identifies with that of degree r differential forms on U. However the structure of the left side of $(*)$ suggests that one should instead consider skewsymmetry *only with respect to rotations* π of the index sequences. This yields a bigger subspace $\Omega^r_{\text{cycl}}(U)$. Now $(*)$ interprets as $\delta X = 0$ where $\delta : \Omega^*_{\text{cycl}}(U) \to \Omega^{*+1}_{\text{cycl}}(U)$ is defined by

$$(\delta X)(\alpha_1\alpha_2\cdots\alpha_{r+1}) =$$

$$\frac{\partial}{\partial\alpha_1}X(\alpha_2\cdots\alpha_{r+1}) \pm \frac{\partial}{\partial\alpha_2}X(\alpha_3\cdots\alpha_{r+1}\alpha_1) \pm \frac{\partial}{\partial\alpha_3}X(\alpha_4\cdots\alpha_{r+1}\alpha_1\alpha_2) \pm \cdots,$$

the signs being as per the same rule as in $(*)$.

It is easily seen that this δ is an extension of the exterior derivative $d : \Omega^*_{\text{alt}}(U) \to \Omega^{*+1}_{\text{alt}}(U)$ and that one has $\delta \circ \delta = 0$. So what is this **cyclic De Rham cohomology** $H^*_{\text{DRcyc}}(U)$? In fact δ extends all the way to $\delta : \Omega^r_{\text{assoc}}(U) \to \Omega^{r+1}_{\text{assoc}}(U)$ if we set

$$(\delta X)(\alpha_1\alpha_2\cdots\alpha_{r+1}) = \sum_i (-1)^i \frac{\partial}{\partial\alpha_i} X(\alpha_i\cdots\widehat{\alpha}_i\cdots\alpha_{r+1}),$$

and one still has clearly $\delta \circ \delta = 0$. So what is this **associative De Rham cohomology** $H^*_{\text{DRassoc}}(U)$? The answers are as follows.

Theorem. *One has* $H^*_{\text{DRassoc}}(U) \cong H^*_{\text{DR}}(M)$ *and*

$$H^*_{\text{DRcyc}}(U) \cong \bigoplus_{j\ge 0} H^{*-2j}_{\text{DR}}(M).$$

Proof of this and some other – e.g. **dihedral De Rham cohomology** is defined similarly by demanding skewsymmetry with respect to rotations and reversals π, and there is a similar formula for it with $4j$ instead of $2j$ – "*associative De Rham Theorems*" will be included in the finalized notes of my seminar of 1994-95. See also the notes of my seminar of 1993-94 for more Poincaré's *Analysis Situs*.

Historical remarks. There had been, off and on, from Zeno to Euler, some contributions to the interplay DISCRETE $\longleftrightarrow$ CONTINUOUS. However it was only in the XIXth century, with the need to understand *discreteness in the theory of integration*, that the work of Riemann, Betti, and Poincaré moved this interplay to mathematical centerstage. An alternative way of defining this "ambiguity of integrals" (§1) without using integrals, is to use (instead of their "integrands" as in §2) their permitted "domains of integration" c. In fact the very first definition of homology given by Poincaré, in §§5-6 of his *Analysis Situs*, is of such a type : it uses singular differentiable chains c and is essentially today's *singular homology.* Later, from §12 onwards of *Analysis Situs*, Poincaré became more combinatorial, and started using a third definition, based only on cellular c's, i.e. essentially today's *usual* (in the next two lectures I'll also be looking at some "very unusual" ones !) *simplicial homology.* For much more on Poincaré's *Analysis Situs* see my seminar notes of 1993-94.

REFERENCES

A. CONNES, *Géométrie Non Commutative*, InterEditions, Paris (1990).

J.-L. LODAY, *Cyclic Homology*, Springer (1992).

K. S. SARKARIA, *Poincaré's Papers on Topology*, notes of Chandigarh Topology Seminar (1993-94).

K. S. SARKARIA, *Combinatorial Methods in Topology*, notes of Chandigarh Topology Seminar (1994-95).

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