Wednesday, July 15, 2009

Problem 1. Let n be a positive integer and let $a_1, \ldots, a_k$ $(k \geq 2)$ be distinct integers in the set $\{1, \ldots, n\}$ such that n divides $a_i(a_{i+1}-1)$ for $i = 1, \ldots, k-1$. Prove that n does not divide $a_k(a_1-1)$.

Problem 2. Let ABC be a triangle with circumcentre O. The points P and Q are interior points of the sides CA and AB, respectively. Let K, L and M be the midpoints of the segments BP, CQ and PQ, respectively, and let Γ be the circle passing through K, L and M. Suppose that the line PQ is tangent to the circle Γ. Prove that $OP = OQ$.

Problem 3. Suppose that $s_1, s_2, s_3, \ldots$ is a strictly increasing sequence of positive integers such that the subsequences

$$s_{s_1}, s_{s_2}, s_{s_3}, \ldots \qquad \text{and} \qquad s_{s_1+1}, s_{s_2+1}, s_{s_3+1}, \ldots$$

are both arithmetic progressions. Prove that the sequence $s_1, s_2, s_3, \ldots$ is itself an arithmetic progression.

Time: 4 hours and 30 minutes
Each problem is worth 7 points

Thursday, July 16, 2009

Problem 4. Let ABC be a triangle with $AB = AC$. The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the sides BC and CA at D and E, respectively. Let K be the incentre of triangle ADC. Suppose that $\angle BEK = 45°$. Find all possible values of $\angle CAB$.

Problem 5. Determine all functions f from the set of positive integers to the set of positive integers such that, for all positive integers a and b, there exists a non-degenerate triangle with sides of lengths

$$a, \; f(b) \; \text{ and } \; f(b + f(a) - 1).$$

(A triangle is *non-degenerate* if its vertices are not collinear.)

Problem 6. Let $a_1, a_2, \ldots, a_n$ be distinct positive integers and let M be a set of $n - 1$ positive integers not containing $s = a_1 + a_2 + \cdots + a_n$. A grasshopper is to jump along the real axis, starting at the point 0 and making n jumps to the right with lengths $a_1, a_2, \ldots, a_n$ in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in M.

An important note for students:--

PLEASE DON"T READ THE SOLUTIONS THAT FOLLOW ….

….*unless* you have already made a very sincere and sufficiently extended effort – say over at least a couple of months – to do these problems by yourself. Otherwise (a) you won't learn much from the terse solutions which follow, and (b) will only succeed in denying yourself that special joy which we all get *only* when we overcome a difficulty all on our own. .

My solutions to IMO 2009 (Bremen) problems

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1. We need to show equivalently that, if two or more distinct integers from $\{1, \ldots, n\}$ are arranged in *circular order* – see Figure 1 – then it is not always the case that n divides the product of two successive integers minus the first.

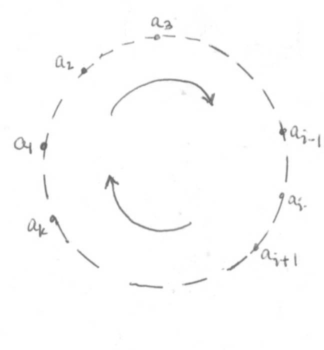

Figure 1

If this were the case, then n would always divide the product of three successive integers minus the first: for, $n|(a_i a_{i+1} - a_i)$ implies $n|(a_{i-1}a_i a_{i+1} - a_{i-1}a_i)$, which implies $n|(a_{i-1}a_i a_{i+1} - a_{i-1})$ because $n|(a_{i-1}a_i - a_{i-1})$. From this it follows by a similar argument that n would always divide the product of four successive integers minus the first, ... , till finally we would obtain that n divides the product of all k integers minus any one of them. So n would divide the differences of these distinct integers $a_i \in \{1, \ldots, n\}$, which is absurd.

2. From K, the midpoint of BP, draw a line parallel to LM, it shall meet BA in its midpoint D – see Figure 2a – likewise, the line LF joining L to the midpoint F of CA is parallel to KM. Let the perpendiculars to MK at K, and to AB at D, meet the right bisector of PQ at $K^\perp$ and $D^\perp$ respectively; likewise $L^\perp$ and $F^\perp$ are the points on this right bisector such that $LL^\perp$ is perpendicular to ML and $FF^\perp$ to AC. *We'll show* $K^\perp D^\perp = L^\perp F^\perp$. This suffices for the result: if the circumcircle of $\triangle KLM$ is tangent to PQ at M, then $K^\perp = L^\perp$ is the intersection of this circle with the right bisector, so the assertion gives us $D^\perp = F^\perp = O$, the circumcentre of $\triangle ABC$, therefore $OP = OQ$.

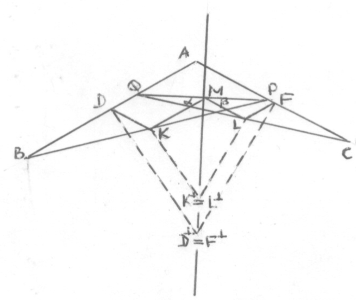

Figure 2a

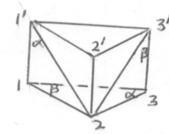

Figure 2b

To prove the assertion we use the associated prism-like Figure 2b which is drawn on a magnified scale: here $\alpha = \angle QMK, \beta = \angle PML$, the horizontals 13 and $1'3'$ are parallel and equal in length (the figure is determined up to congruence by this length and the angles α, β) to $QM = MP$, and the verticals are parallel to the right bisector of QP. Clearly 12 is equal and parallel to DK, and 23 to LF. Then, since $\angle 1'23 = 90°$, $21'$ is parallel to $D^\perp D$, and it follows that the length of the vertical $11'$ is the same as that of $D^\perp K^\perp$. Likewise, since $\angle 1'23 = 90°$, $23'$ is parallel to $F^\perp F$ and the vertical $33'$ has the same length as $F^\perp L^\perp$. Thus both these lengths are equal, and equal to that of $22'$.

3. Given any *jump* $s(n)$ to $s(n+1)$ of the strictly increasing sequence, the $t = s(n+1) - s(n)$ successive jumps – see Figure 3 – of the sequence, as the index moves from $s(n)$ through $s(n+1)$, shall be called its *secondary jumps.* Since these take us from $s^2(n)$ to $s^2(n+1)$, secondary jumps always add up to the common difference a of the arithmetical progression $s^2(n)$.

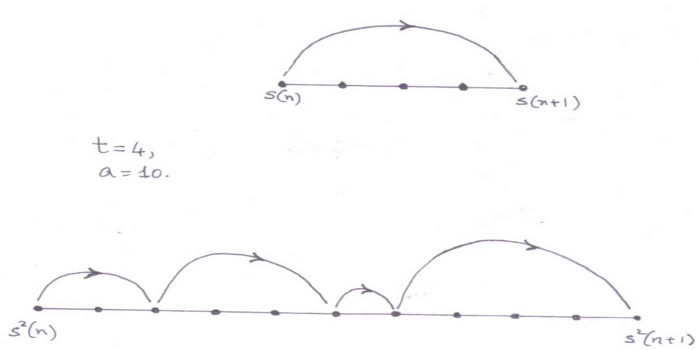

Figure 3

Further, if the size t of our jump has the minimum possible value m, then all its m secondary jumps have the maximum possible value M. Otherwise, there is a jump of size bigger than a/m, and the average size of the secondary jumps of this jump shall be less than $a \div a/m = m$, contradicting the minimality of m. Likewise, if the size t of our jump has the maximum possible value M, then all its M secondary jumps have the minimum possible value m.

If $s(s(n)+1)$ is also an arithmetical progression, then it has the same common difference a, for, $s(n) < s(n)+1 \leq s(n+1)$ implies $s^2(n) < s(s(n)+1) \leq s^2(n+1)$, i.e., this sequence alternates with $s^2(n)$. Let c denote the constant phase difference $s(s(n)+1) - s(s(n))$ between the two arithmetical progressions. In case the jump from $s(n)$ to $s(n+1)$ has the minimum size m, we see from the last paragraph that $s(s(n)+1) - s(s(n)) = M$, while $s(s(n)+1) - s(s(n)) = m$ if the said jump has the maximum size M. Thus this additional hypothesis can hold iff $m = M = c$, that is, iff $s(n)$ is itself an arithmetical progression.

4. We assume $BC = 2$, take its mid-point D as the origin $(0,0)$, and choose the positive x- and y-axes to be along DC and DA, respectively. If the incentre I has coordinates $(0,t)$, the coordinates of the other pertinent points can be easily worked out, and are indicated in Figure 4. For example, since t is the tangent of half the base angle B, the y-coordinate of A, i.e. $\tan B$, must be $\frac{2t}{1-t^2}$; then, the coordinates of E, the point on AC such that BE has slope t, work out to be $(\frac{1+t^2}{3-t^2}, \frac{4t}{3-t^2})$; and those of K, the point on IC with equal coordinates, turn out to be $(\frac{t}{1+t}, \frac{t}{1+t})$; etc.

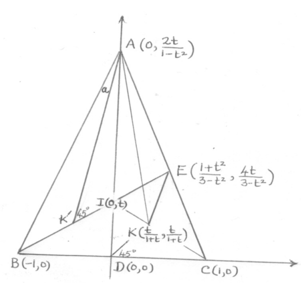

Figure 4

If a is one-fourth the angle at the vertex A, then $t = \tan(45° - a)$. Also note that one always has $\angle AK'E = \angle ABK' + \angle BAK' = (45° - a) + a = 45°$. Therefore the isosceles triangle ABC satisfies the required condition $\angle BEK = 45°$ if and only if AK' is parallel to EK, i.e., AK' and EK have the same slope, i.e., $(\frac{2t}{1-t^2} - \frac{t}{1+t}) \div \frac{t}{1+t} = (\frac{4t}{3-t^2} - \frac{t}{1+t}) \div (\frac{1+t^2}{3-t^2} - \frac{t}{1+t})$, i.e., $3t^4 + 6t^3 - 4t^2 - 2t + 1 = 0$, i.e., $(3t^2 - 1)(t^2 + 2t - 1) = 0$. The positive solutions of this equation are $t = \sqrt{2} - 1 = \tan 22.5°$ and $t = 1/\sqrt{3} = \tan 30°$, giving $a = 22.5°$ or $15°$, i.e., the triangle must be *right isosceles or equilateral.*

5. Each side of a triangle is less than the sum of the other two. When $b = 1$ the sides are $\{a, f(1), f^2(a)\}$ which shows $a - f(1) < f^2(a) < a + f(1)$, in particular that f cannot be bounded. For $a = 1$, the triangle with integral sides has one side of minimum length 1, so its other two sides are equal, i.e., $f(b) = f(b+f(1)-1)$. This shows $f(1) = 1$, for otherwise, f would be periodic, so bounded. Hence we have $f^2(a) = a$ for all a, that is f is a bijection of the positive integers which is its own inverse: if $a' := f(a)$, then $a = f(a')$.

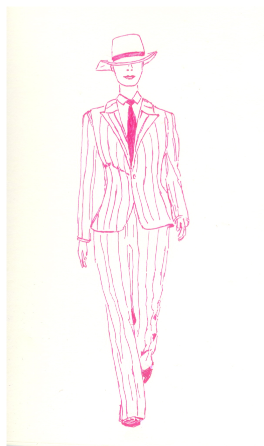

Figure 5

So, if we use b' in place of b, the triangle has sides $\{f(a'), f(b'), f(a'+b'-1)\}$, which shows that f satisfies the inequality $f(a'+b'-1) \leq f(a')+f(b')-1$ for all pairs of integers. It follows inductively that $f(1+(r-1)(2'-1)) = r$ for all $r \geq 1$: $f(1+(r-2)(2'-1)+2'-1) \leq f(1+(r-2)(2'-1))+f(2')-1 = (r-1)+2-1 = r$, and therefore, equal to r. If $2'$ were bigger than 2 we have a proper subset imaging under f to the whole set of positive integer. So we must have $2' = 2$ and $f(n) = n$ for all n, i.e., f *is the identity function.*

[For this solution I couldn't think of a 'suitable figure', for example, a diagram of a generic triangle with sides labelled a, $f(b)$ and $f(b + f(a) - 1)$ would have been, oh so trite, and only a trifle more illuminating than Figure 5!]

6. For $n = 1$ and 2 the assertion is obvious, so assume inductively its truth for all values lesser than a given $n \geq 3$. Let α be the biggest member of the given cardinality n set A of permitted jumps, and μ the smallest member of an arbitrary cardinality $n-1$ set M of forbidden landings.

If α is not in M or $\alpha = \mu$, we can use the inductive hypothesis for $n-1$ to make the $n-1$ jumps other than α in a suitable order, say $\alpha_1, \ldots, \alpha_{n-1}$, to go from α to s without landing on the $n-2$ points of M other than μ. Let α_j denote the jump from μ in this process, and if there is no such jump put $j = 0$. Then the sequence of jumps, $\alpha_1, \ldots, \alpha_j, \alpha, \alpha_{j+1}, \ldots \alpha_{n-1}$ – see Figure 6 – shall take us from 0 to s without landing on any point of M.

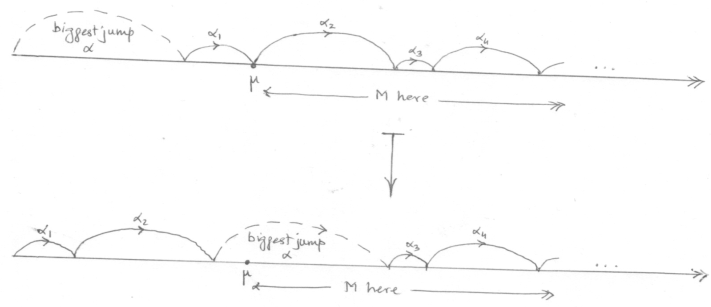

Figure 6

Otherwise, let k denote the number of points of M less than or equal to α, and note that $2 \leq k \leq n-1$. There are at least $n-k$ jumps a of A with a not in M, and since the corresponding numbers $a+\alpha$ are distinct, we can choose from these one for which $a+\alpha$ is different from the $n-1-k$ points of M bigger than α. Starting from 0, we first make this jump a, and then α; this successfully clears at least two points of M; then, by invoking the inductive hypothesis for $n-2$, we use the remaining $n-2$ jumps in a suitable order to go from $a+\alpha$ to s without landing on the, at most $n-3$, remaining points of M.

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