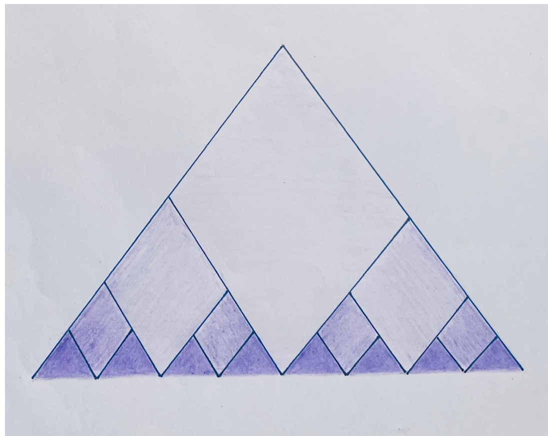
... but nowhere differentiable!

K. S. Sarkaria



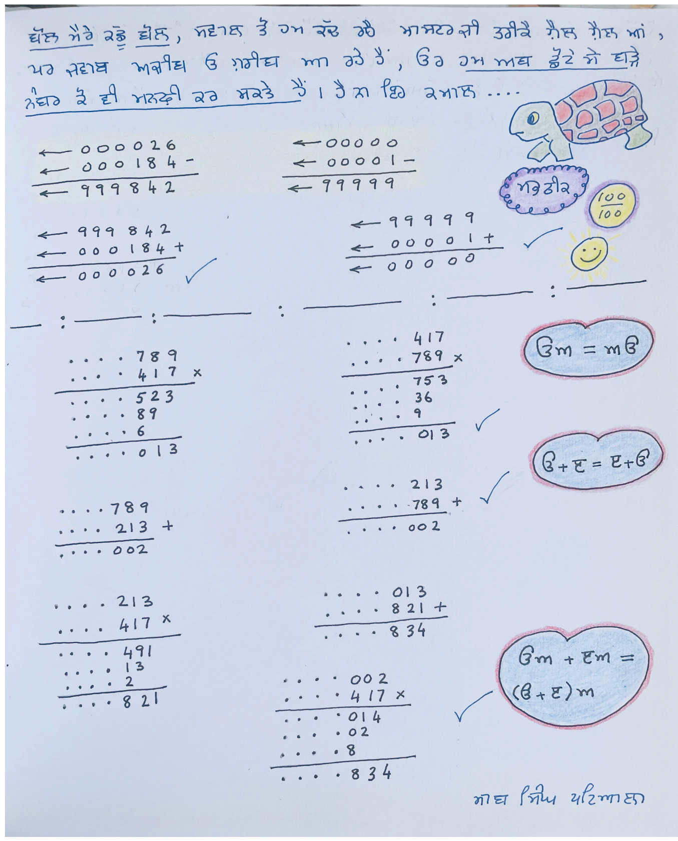
About sixty years ago I discovered, much to my delight, that continuous but nowhere differentiable functions are in fact quite easy to visualize.[1] Given any triangle, any two of its sides give us the graph, over a period, of a continuous periodic function with period equal to the length of the third side. Our x-axis here is along the third side, and the (possibly very oblique) y-axis parallel to the median from the opposite vertex. Now draw parallels to the other two sides from this midpoint to get two similar triangles of half the size and iterate this construction. The sum of this infinite sequence of continuous functions, each with period half that of the previous, is continuous because it is dominated by a geometric series of constants with common ratio half.

We can assume the equal but opposite slopes of the two sides ± 1 (and even put a euclidean metric such that the axes become perpendicular). To see that the sum function is nowhere differentiable we nest any given x for each m in a closed interval between consecutive zeros of its mth summand. The slope of the chord over this interval for this and all subsequent functions is zero, while for all previous functions in our sequence it is $+1$ or -1; so the slope of the chord of our sum function is alternately an odd or even integer for m even or odd. So it does not approach a finite limit as m goes to infinity. $\Box$

Maya <u>is</u> duality :- *quotients of all sequences of a dual* $\{0, 1\}$ *give us about all the shapes we see!*[8] Only, the identifications needed to create a hat or an elephant or a homology 3-sphere in 4-space, are a *wee* more involved than the familiar base two identifications of eventually 1 with eventually 0 sequences that create a segment, say the base of our triangle.

Using instead of $\mathbf{2} = \{0, 1\}$ sequences from a bigger finite set $\mathbf{g}$, say the digits $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ of base ten notation, gives us no new shapes, because the product spaces $\mathbf{2}^{\mathbb{N}}$ and $\mathbf{g}^{\mathbb{N}}$ are homeomorphic, but it gives us *a whole panoply of distinct arithmetics in this cantorian dust!*

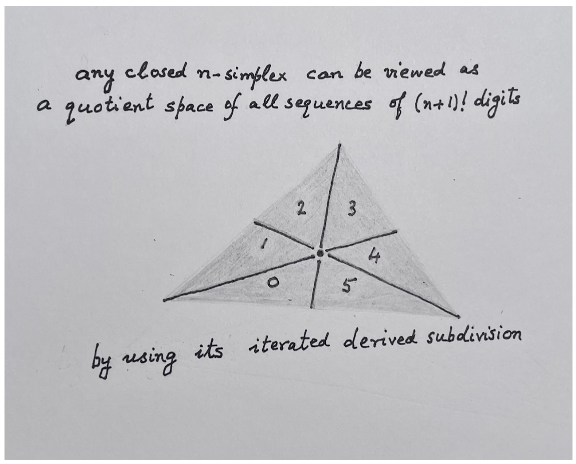
I have it on good authority that *these arithmetics were discovered by Sahib Singh of Patiala.* The primary school work of this royal prodigy–certified copies of the original folios in the archives are awaited–show clearly (for we still use the same Arabic numerals) that he was in full possession of *the ring* $\mathbb{Z}_{10}$ *of* 10-*adic integers modulo jargon* at this tender age:-



It seems after learning the usual carry operations on finite sequences of digits (whole numbers) this child had one fine day the great idea that if he wrote them as (eventually zero) *infinite left sequences,* then he could subtract even bigger numbers from smaller! The rest, the carry operations extend to all infinite left sequences and obey the same laws, was easy. □

Infinite sequences of digits had sneaked up on an older me in 'boring' decimal calculations. That how utterly magical their underlying idea in fact was dawned on me years later! In hindsight there was no reason for this delay. We created *from the dust of ten digits a segment*:- the first digit picks from ten equal parts that in which a point lies, the second from hundred equal parts that in which a point lies, etc. Clearly by dividing successively into $g, g^2, g^3 \ldots$ equal parts we *can use here instead of ten any whole number* $g \neq 1$. Likewise (even from the discrete dust of 2 digits) we can make an *n-simplex,* say by repeatedly deriving it; so any *simplicial complex*, so all sorts of *manifolds* in our three or even in more dimensions; and baby we have just begun . . . □

Those pesky identifications .42000... = .41999..., etc., of decimals are but the interior vertices of this iterated subdivision of $[0,1]$: these are intersections of nested intervals of length $\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \ldots$ in exactly two ways, all other points of the segment in a unique way; and likewise for any base g. Yet *more glue* needs to be applied to this compact discontinuum to view *higher dimensions,* for example, as we create a closed 2-simplex, that is a triangle, by using its iterated derived subdivision, then, for each term, over some points - see figure - of this closed sub 2-simplex, there is a two or six fold ambiguity regarding the next term, for they are incident to as many triangles of its derived. However, way back in the day then, even those pesky identifications had become clear only after I learnt about infinite geometric series, when they became $4 \times 10^{-1} + 2 \times 10^{-2} = 4 \times 10^{-1} + 1 \times 10^{-2} + 9 \times 10^{-3} + 9 \times 10^{-4} + \cdots$, etc.

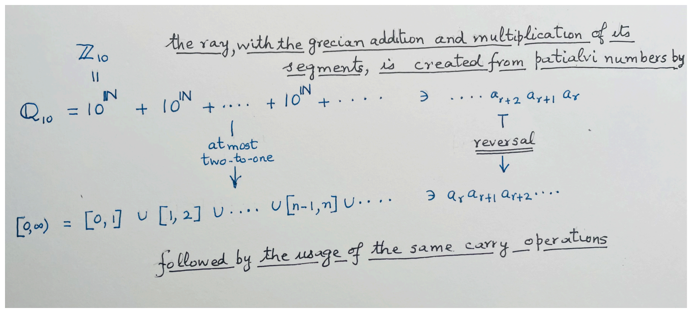


Using its iterated base g subdivision, $[0.\infty)$ can be viewed as a quotient space of a sum (disjoint union) of denumerably many compacta $\mathbf{g}^{\mathbb{N}}$ above respectively the segments $[0,1], [1,2], \ldots$ that concatenate to make this *ray.* Each point of the ray other than the nonzero vertices of subdivision has just one pre-image, these have two. Applying more glue on this locally compact discontinuum $\mathbb{Q}_g$ we get also higher dimensional locally compact manifolds, etc.

These g-adic numbers $\mathbb{Q}_g \supset \mathbb{Z}_g$ are all left sequences $\ldots, a_{r+2}, a_{r+1}, a_r \neq 0$ of g digits where now the first *place* $r \in \mathbb{Z}$ with a nonzero term may not be in the positive integers $\mathbb{N} \subset \mathbb{Z}$; addition and multiplication by usual base g carry operations makes $\mathbb{Q}_g$ a bigger ring, and even a field if g is prime.

The *reversal* $\ldots, a_{r+2}, a_{r+1}, a_r \mapsto a_r \frac{1}{g^r} + a_{r+1} \frac{1}{g^{r+1}} + a_{r+2} \frac{1}{g^{r+2}} + \cdots$ or briefly $a_r, a_{r+1}, a_{r+2}, \ldots,$ gives the point below of the created ray; so the condition, integral part after reversal is $n-1$, prescribes the summand $\mathbf{g}^{\mathbb{N}} \subset Q_g$ creating $[n-1, n]$. Also, *reversal creates from patialvi arithmetic grecian addition and multiplication* of initial segments of the ray[10] : for these are given to any desired approximation by the same carry operations. $\square$

Mimicking this (using the same carry operations on eventually zero sequences approximating the two points of the quotient to be added or multiplied) does not give well-defined binary operations in general, and even for a ray, though its *arithmetic as a process* is essentially the same as upstairs, clearly the quotient map $\mathbb{Q}_g \downarrow [0, \infty)$ does not commute with their outputs, for example, no nonzero point of the ray has an additive inverse in it.



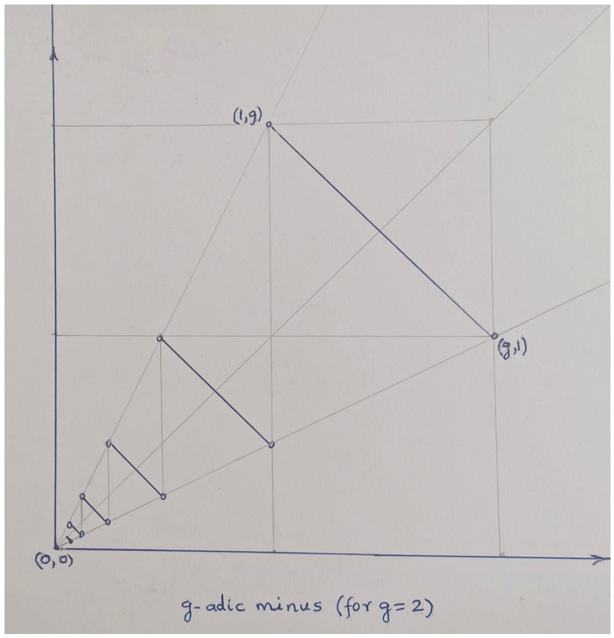
The usual natural *lifting problems* arise: which functions in the quotient space lift up, which functions upstairs are lifts, and if continuity, differentiability, etc., is preserved ? We recall the operations of $\mathbb{Q}_g$ are continuous in its topology, and the *derivative* of a function f whose domain and range are in it is once again $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ whenever r.h.s. makes sense and exists. Occasionally there are also interesting answers.

The additive inverse of $\mathbb{Q}_g$ lifts the order reversing linear isomorphisms of the components of $[0, \infty) \setminus \{g^r, r \in \mathbb{Z}\}$:- Let's first see why the powers of the base, say 10, need to be excluded. Take, e.g., $10 = 9.999 \cdots \in [0, \infty)$ below the 10-adic numbers $\ldots 0001$ with first nonzero place or *value* $r = -2$ and $\ldots 9999$ for which $r = -1$. To subtract them from $\ldots 0000 \in \mathbb{Q}_{10}$ we put them below it so places above and below match and use carry operations: this gives respectively $\ldots 9999$ with value $r = -2$ and $\ldots 00001$ with value $r = -1$, which lie above distinct points $99.999 \cdots = 100$ and 1 of the ray.

Clearly 10-adic minus is a lift above points of the ray below a unique 10-adic number, that is below left sequences of the digits neither eventually 0 nor 9, for this remains true after subtraction from $\ldots 0000 \in \mathbb{Q}_{10}$.

Above the remaining countable but dense subset of the ray also 10-adic minus is a lift. Take, e.g., $25.36 = 25.35999 \cdots \in (10, 100)$. The two 10-adic numbers above it are $\ldots 0006352$ and $\ldots 9995352$ both with $r = -2$. Subtracting them from $\ldots 0000 \in \mathbb{Q}_{10}$ we get respectively $\ldots 9993648$ and $\ldots 0004648$ both with $r = -2$. So both above the point $84.63999 \cdots = 84.64$ of $(10, 100)$ which is at the same distance from the end 100 of this interval as 25.36 was from 10. Likewise this involution below pairs points equidistant the two ends, in particular the mid-point $55 = 54.999 \cdots$ is a fixed point of this involution because the two 10-adic numbers above it $\ldots 00055$ and $\ldots 99945$ both with $r = -2$ are merely interchanged when we subtract them from $\ldots 0000 \in \mathbb{Q}_{10}$. $\square$

This involution of $[0, \infty) \setminus \{g^r, r \in \mathbb{Z}\}$ is drawn below for $g = 2$, but note this symmetric binary relation on the ray, that is, subset of $[0, \infty) \times [0, \infty)$, varies continuously over *all real* $g > 1$. It is tempting to muse over what this means when g is not a whole number, because the iterated subdivision of the ray into closed subintervals of length $\frac{1}{g}, \frac{1}{g^2}, \ldots$ now does not admit like carry operations, but there may be a noncommutative interpolation of these for numbers between $g = 1, 2, 3, 4, \ldots$; however we'll not pursue this here because we have enough and to spare on our plate already.



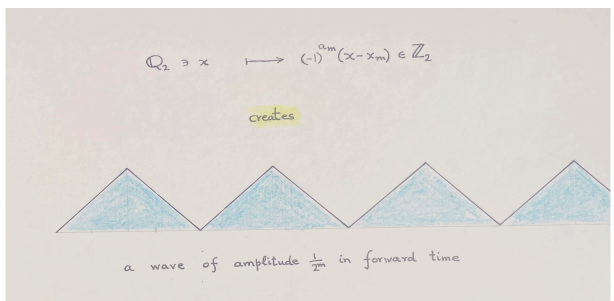
We note the above function is differentiable with constant derivative -1 on each open intervals of its domain, just like the function $x \mapsto -x$ creating it which is defined on all of $\mathbb{Q}_2$ and has constant derivative -1. For any prime base p, the product of two nonzero digits being nonzero mod p, we can not only talk of division by numbers not zero, and so of *rational functions in* $\mathbb{Q}_p$ (whose domain lacks only the finitely many zeros of the denominator) but also we know these functions are all differentiable with derivatives given by the usual formulas. Of these the few which lift functions defined on at least an open interval of the ray $[0, \infty)$ deserve now to be worked out, to see for example if there is correlation between derivatives above and below; but beyond algebraic functions tied to the five p-adic operations there are many others ...

Dust thou art to dust returnest: *summands f_m of our continuous but nowhere differentiable function lift over $[0, \infty)$ to functions $\tilde{f}_m : \mathbb{Q}_2 \to \mathbb{Z}_2$ whose sum is also continuous but nowhere differentiable*:-

We recall $f_m : \mathbb{R} \to [0, 1]$ has period 2^{1-m} with $f_m(x) = x$ if $0 \le x \le 2^{-m}$ and $f_m(x) = 2^{1-m} - x$ if $2^{-m} \le x \le 2^{1-m}$. So f_m maps $x \in [0, \infty)$ with base 2 expansion $x = a_{-t} \cdots a_0.a_1a_2 \cdots$ to $.0 \cdots a_m a_{m+1} \cdots \in [0, 1]$ if $a_m = 0$ and (we use overlining for $0 \leftrightarrow 1$) to $.0 \cdots \overline{a}_m \overline{a}_{m+1} \cdots \in [0, 1]$ if $a_m = 1$.

So f_m lifts to $\tilde{f}_m$ which maps any 2-adic number $x = \ldots a_2a_1a_0 \ldots a_{-t}$ to the 2-adic integer $\ldots a_{m+1}a_m$ if $a_m = 0$ and to $\ldots \overline{a}_{m+1}\overline{a}_m$ if $a_m = 1$. Only the *dyadic rationals* $x \in [0, \infty)$, i.e., vertices of the iterated base two subdivision, have two base two expansions, that eventually 0 reversed we denote by $x_+ \in \mathbb{Q}_2$, the other by $x_- \in \mathbb{Q}_2$; the 2-adic number above other $x \in [0, \infty)$ will also be denoted $x \in \mathbb{Q}_2$. We note that f_m maps the dyadic rationals and only these to dyadic rationals $f_m(x)$, and that then upstairs $\tilde{f}_m$ maps the pair $x_\pm$ onto the pair $f_m(x)_\pm$ though switching can occur.

Let's dub the eventually zero number $x_m = \ldots 00a_m \ldots a_{-t} \in \mathbb{Q}_2, m \geq 1$ the *mth truncation* of $x = \ldots a_2a_1a_0 \ldots a_{-t} \in \mathbb{Q}_2$. We note that $\tilde{f}_m(x) = x - x_m$ if $a_m = 0$ while $\tilde{f}_m(x) = x_m - x$ if $a_m = 1$; further, all 2-adic numbers having this same mth truncation $c = x_m$ form a clopen subset of $\mathbb{Q}_2$. So for all 2-adic numbers y in this neighbourhood U_c of x we have always $\tilde{f}_m(y) = y - c$ or else $\tilde{f}_m(y) = c - y$. So, using the partition $\mathbb{Q}_2 = U_{m.0} + U_{m,1}$ into clopen sets of all $x \in \mathbb{Q}_2$ with $a_m = 0$ or 1, *$\tilde{f}_m : \mathbb{Q}_2 \to \mathbb{Z}_2$ has constant derivative $+1$ on $U_{m,0}$ and -1 on $U_{m,1}$*, while the created $f_m : [0, \infty) \to [0, 1]$ is not differentiable at some isolated points but elsewhere has derivative ± 1 :-

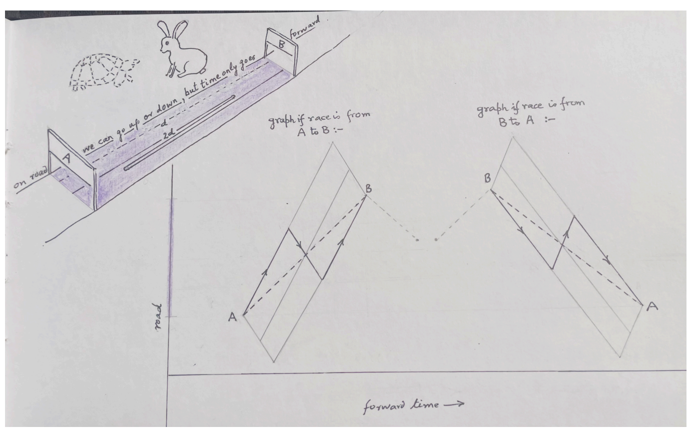


So counting turning points twice, $U_{m,0} \subset \mathbb{Q}_2$, all 2-adic numbers with mth digit $a_m = 0$, is above the part of $[0, \infty)$ where this wave has one-sided derivative $+1$, each interval contributing a summand U_c on which mth truncation $x_m = c$; while its complement $U_{m,1}$ is the disjoint union of the remaining U_c above the intervals where this wave has one-sided derivative -1. Further, the graphs in $\mathbb{Q}_2 \times \mathbb{Z}_2$ of the lifts $\tilde{f}_m$ project to those of these waves f_m in $[0, \infty) \times [0, 1]$. But, the graph of the patialvi sum of two or more of these lifts is not above the graph below of the grecian sum of the corresponding waves.

Anyway, we know by now these finite sums, say $\tilde{f}_1 + \cdots + \tilde{f}_N : \mathbb{Q}_2 \to \mathbb{Z}_2$ are differentiable everywhere, with derivative $-N$ where all summands have derivative -1, derivative $-N + 2$ where exactly one has derivative $+1$, ... , and derivative $+N$ where all summands have derivative $+1$. As for the infinite sum $\sum_{m \geq 1} \tilde{f}_m(x)$, we note that this series converges uniformly because the value of $\tilde{f}_m(x)$ is bigger than m for all $x \in \mathbb{Q}_2$. Therefore its sum $R(x)$ is continuous. If a derivative $R'(x), x \in \mathbb{Q}_2$ exists at all, it certainly is not $\sum_{m \geq 1} \tilde{f}'_m(x)$: this series diverges for all x because its terms are ± 1.

To clinch the nonexistence of $R'(x)$ we compute the slopes $\frac{\Delta R}{\Delta x}$ for increments $\Delta x = 1_N$, the 2-adic integer with all digits 0 except 1 at the Nth place, or $\Delta x = -1_N$ which has 1's only at all places from the Nth leftwards, and check that the successive differences of these slopes as N jumps to $N+1$ do not converge to zero. Adding $\pm 1_N$ to $x = \ldots a_2 a_1 a_0 \ldots a_{-t} \in \mathbb{Q}_2$ does not alter its mth truncation if m is less than N so then $\frac{\Delta \tilde{f}_m}{\Delta x} = \pm 1$. So the sum of these slopes jumps by ± 1. The remainder $R_N(x) = \tilde{f}_N(x) + \tilde{f}_{N+1}(x) + \cdots$ has 0 digits only up to the Nth place. We'll give to x the increment 1_N if $a_N = 1$ or the increment -1_N if $a_N = 0$ to ensure that the pair of digits at the Nth and $(N+1)$th place stay alike or unlike as the case might be. So the digit at the $(N+1)$th place of $R_N(x)$, which is the same as the digit at this place of $\tilde{f}_N(x)$, stays put. Thus the first nonzero digit of the left sequences $\pm \Delta R_N(x)$ is encountered at the $(N+2)$th place or later. Division by $\Delta x = 1_N$ is the same as right shift by N places. So $\frac{\Delta R_N(x)}{\Delta x}$ has its first nonzero digit only at the 2th place or later. Thus the successive differences of $\frac{\Delta R}{\Delta x}$ as N jumps to $N+1$ have digit 1 in their first place and don't converge to zero. $\square$

If forward time is itself periodic 'this all' is going to be **reincarnated** ad infinitum! So, more humbly, we turn to graphs showing displacements of things as time elapses, since relative motion, periodic or not, is something we do see in this all. *Continuous but never differentiable motion* was attained by a kind, bold and devout bohemian *by using ad infinitum* this grecian paradox: *'a tortoise covers a certain length at a steady speed, a hare does the same in the same time but moving always at twice the speed'*:-



Besides covering the given segment in the same direction as the tortoise, the hare must run an equal extra amount for half the given time. This we'll allow him to do by running an even number of times more some interval(s) of the segment itself (or even beyond its ends on that line). Objections like how could he reverse instantly up to the same speed we'll ignore; indeed we plan to iterate, i.e., *the hare will become a tortoise to a second hare twice as fast as him, and so on and on,* so our tortoises and hares shall soon be flying much faster than the speed of light in this flight of pure fancy! Of the uncountably many allowed solutions we have shown above the simplest: the hare dashes ahead to $\frac{3}{4}d$, then reverses past the tortoise at mid-point $\frac{1}{2}d$ back to $\frac{1}{4}d$, where he reverses again to finish the race in the exact same time 1 as the tortoise.

Graphically the progress in time $0 \le t \le 1$ of the tortoise is that dotted segment with a nonzero slope s. It is replaced by the graph ϕ_1 of the hare which has *three segments: the first and the third on parallel lines of slope $2s$ through the ends while the second is on the line of slope $-2s$ through the mid-point.* This in turn is replaced by the graph ϕ_2 of the second hare which has nine segments with slopes alternately $4s$ and $-4s$, obtained by applying the same construction to each of the three segments of ϕ_1; and so on; thus the graph ϕ_n of the nth hare has 3^n segments of slopes alternately $2^n s$ and $-2^n s$. Since the absolute value of the rise or fall of each of the three segments is at most three-fourth that of the segment on which we apply our construction, $|\phi_{n+1}(t) - \phi_n(t)| < (\frac{3}{4})^n d$, which shows that all these hare-y motions converge uniformly as n goes to infinity to a continuous but obviously very hairy motion $\phi(t), t \in [0,1]$.

All vertices–also mid-points of segments–of any ϕ_n are on ϕ_{n+1}, so on ϕ, so *all segments of any ϕ_n are chords of ϕ*; their slopes being large in absolute value for n big, but positive or negative alternately, on this countable dense subset of $(0,1)$ no finite or infinite value can be assigned to $\phi'(t)$, but $\phi'(0) = \phi'(1) = \pm\infty$ depending on whether s is positive or negative.

No single finite or infinite value can be given to $\phi'(t)$ at any other $t \in (0,1)$ either. To see this note that now, for each n, there is a unique segment P_nQ_n of ϕ_n such that P_n *comes before $\phi(t)$ and Q_n after it* in this limiting motion. Were $\phi'(t)$ to exist the slopes of these ever smaller *straddling chords* P_nQ_n should also approach this value as n goes to infinity. So certainly no finite value can be assigned to $\phi'(t)$, but what if the slopes of all but finitely many of these ever steeper chords are positive, or else almost all are negative? Then we'll again use, respectively, the two graphs above with a few changes.

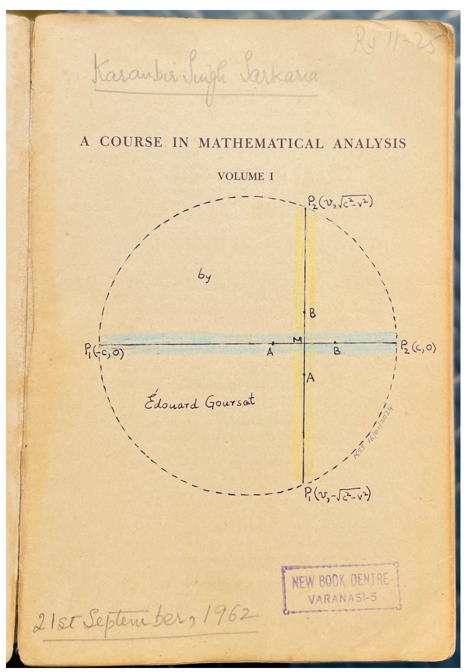
If almost all these slopes are positive we use the first graph with AB deemed to be P_nQ_n where n is big. So big that one of the positive sloping segments of the *zee* obtained by performing our construction on P_nQ_n is $P_{n+1}Q_{n+1}$. Accordingly we label the zee $P_n = P_{n+1}, Q_{n+1}, R_{n+1}, Q_n$ or $P_n, R_{n+1}, P_{n+1}, Q_{n+1} = Q_n$ and extend the dotted graph ϕ_n leftwards or rightwards–only this case is drawn–till the mid-point S_n of the adjacent segment. We claim that the straddling chord of $\phi(t)$ joining R_{n+1} and S_n has a large negative slope. To verify this for the case drawn, let S_n have coordinates (h_n, k_n) with respect to P_n, then the coordinates of Q_n are $(\frac{3}{4}h_n, \frac{3}{2}k_n)$, so those of R_{n+1} are $(\frac{3}{4}\frac{1}{2}\frac{3}{4}h_n, \frac{3}{2}\frac{1}{2}\frac{3}{2}k_n)$, from which we see that S_nR_{n+1} has slope $-\frac{4}{23}\frac{k_n}{h_n}$; and a similar calculation gives the same answer for the other case. If almost all P_nQ_n have negative slope we use likewise the left↔right mirror image, i.e., the second graph. □

So, *arbitrarily close to any continuous function there's another which is also nowhere differentiable*:- tortoises ↔ hares play on each segment of a piecewise linear approximation. □ Also, *they can play many ways,* e.g., the hare's motion in time can be on equally spaced lines of slope $2s$ or $-2s$, for, if mesh is small, most of these *lattice paths* keep hare within track; indeed as long as, *each leg run is shorter than the track,* we can allow him to even overrun either end:- iterating as before gives a continuous motion from A to B in the same time which is never differentiable. □ *We can likewise iterate ad infintum* **any** *piecewise linear hare-y* *motion within the track* – or even let him overrun the track as long as each leg run is shorter than the track – and it seems likely that:- *if our rabbit doesn't nap, and the total distance he runs is more than his pal, then this limiting continuous motion is never differentiable* **?**

This cliffhanger seems just the place from which to continue these musings, as has been my wont in the past too, in the guise of *Notes,* of which only some will be annotations, others reminiscences, but hopefully there shall be also again eventually many more theorems, proofs and problems below than above, but be ready as well for irrelevant but insistent thoughts, etc., so with this understood to be the very wide scope of this word here, now come these

Notes.

1. All I 'discovered' really was just the beautiful Chapter I, *Differentiation,* of Riesz and Sz.-Nagy (1952). The rest—this was in 1966—drawing pictures to follow say its first two pages and making two triangles of half the size instead of ten of one-tenth, etc., was easy and fun. I'm not sure if I still have those sketches and stuff, but the joy of understanding the discoveries of Lebesgue, Denjoy et al., is still vivid. For example the picture in *Distances and homeomorphisms* was also drawn from memory from that time only.



2. At this point I'll urge you to look at probably my best paper PG&R (2013-16), of which the best is in some of its later notes, but just from its note 5 you can learn in quick order how I 'discovered' mathematics in 1962 and why *Cours d'analyse mathématique* (1902) by Goursat is special to me.

Its hand-drawn 'frontispiece' is however only a few months old.[1] Using it and $\log \frac{1+x}{1-x} = 2(\frac{x}{1} + \frac{x^3}{3} + \frac{x^5}{5} + \cdots)$ on page 102 of this volume I checked that *the limit of* $\log \frac{P_1B}{P_1A}\frac{P_2B}{P_2A} \div AB$ *as* AB *shrinks to its mid-point* M is $g_1(v) = \frac{2c}{c^2-v^2}$, respectively, $g_2(v) = \frac{2}{\sqrt{c^2-v^2}}$, for the two cases shown in this frontispiece, i.e., when the element AB is radial, respectively, tangential.

A reflection-invariant[2] *riemannian metric on an open ball* of dimension two or more is *plain* : it has the same polar form $(g_1dr)^2 + (g_2rd\theta)^2$ on any plane section through its centre, $g_1(r)$ and $g_2(r)$ being its infinitesimal radial and tangential comparison ratios. The above calculation of limits shows that the *cayley distance*—see PG&R—on an open ball is such a riemannian metric, viz., that with $g_1(v) = \frac{2c}{c^2-v^2}$ and $g_2(v) = \frac{2}{\sqrt{c^2-v^2}}$ where c is the radius.

3. On page 6 already Goursat informed me that Weierstrass gave 'examples of continuous functions which do not possess derivatives for any values of the variable whatever' with a footnote citing his 1872 note and a memoir by Darboux with other examples; and also informed me that 'one of Weierstrass's examples is given later (Chapter IX)': it is way later, § 200, pp. 423-425 [3] but I must have worked my way to and through it by 1964.

As usual *differentiable means has a unique finite derivative,* page 7, and on page 6 about $y = x^{\frac{2}{3}}$ Goursat says its 'derivative is said to be infinite for $x = 0$' though the difference quotient $y/x = x^{-\frac{1}{3}}$ approaches $+\infty$ from the right and $-\infty$ from the left; accordingly–page 10–'if two points be taken on the curve $y = x^{\frac{2}{3}}$, on opposite sides of the y axis, it is evident from a figure' the slope of the line joining them can approach any limit whatsoever 'by causing the two points to approach the origin according to a suitably chosen law': take a secant of the required slope and slide it parallely towards the origin. □ However, *if a unique derivative* $f'(x)$*, finite or infinite, exists, then* $\frac{f(x+h)-f(x-k)}{h+k}$ *approaches* $f'(x)$ *irrespective of how* $h > 0$ *and* $k > 0$ *approach zero*:- because this slope is a convex combination $\frac{h}{h+k}\left(\frac{f(x+h)-f(x)}{h}\right) + \frac{k}{h+k}\left(\frac{f(x)-f(x-k)}{k}\right)$ of difference quotients from the right and from the left, which are in an arbitrary convex neighbourhood of $f'(x) \in [-\infty, +\infty]$ provided h and k are sufficiently small. □

All alone I'd worked out such points raised in the text, not omitting at all articles like §200 in smaller print, indeed these I'd found often more exciting, e.g., the preceding §199 containing the beautiful proof of Lebesgue [4] of Weierstrass's polynomial approximation. However the initial resolve to proceed to the next chapter only after resolving all exercises melted with *an exasperating exercise* given at the very end of the very first chapter:-

[1] This all is from unposted note 20 of MGH, but I'd posted asap note 22.

[2] I.e., preserved by all reflections preserving the ball, i.e., $O(n-1)$-invariant.

[3] Trifling changes here–keep track of $h < 0$ and $0 < h$ separately–show it has no unique infinite derivative either at any point.

[4] A pencilled note here–of much later vintage, then I had no access to journals–says this 1898 paper was Lebesgue's first. As Riesz and Sz.-Nagy made abundantly clear to me later, *the idea of Lebesgue measure and integral was born out of a desire to get around nowhere differentiable continuous functions.* More immediately, by the end of §199 it was clear that, *to grasp the general ideas and results of real analysis humble piecewise linear examples suffice,* but of course, one must also draw lots and lots of figures.

19*. *The* n*th derivative of a function of a function* $u = \phi(y)$, *where* $y = \Psi(x)$, *may be written in the form* $D_x^n \phi = \sum \frac{n!}{i!j!\cdots k!} D_y^p \phi(\frac{\Psi'}{1})^i(\frac{\Psi''}{1.2})^j(\frac{\Psi'''}{1.2.3})^k \cdots (\frac{\Psi^{(\ell)}}{1.2\ldots\ell})^k$, *where the sign of summation extends over all the positive integral solutions of the equation* $i + 2j + 3h + \cdots + \ell k = n$, *and where* $p = i + j + \cdots + k$.

An inordinate amount of time—I may still have those messy pages—yielded *an unenlightening verification* but this pyrrhic victory taught me (more later) that even in mathematics there is room for this sage precept: *"Proceed! and faith* = understanding *will come unto you."* Moreover I had skimmed lightly over even the text of the last three chapters–applications to geometry of plane curves, skew curves and surfaces–because I had learnt the basics from other books, but above all because the next volume awaited.

4. Chapters I-V of Goursat, vol. II, revel in the beauties of *complex functions of a complex variable that are differentiable.* Over $\mathbb{C}$ differentiability is tight, and tied to two-dimensional heat conduction, using a boundary value solution of which was found the most striking of all these beauties, *Riemann's theorem* (1851): this is stated in §22 of Ch. I with some nice examples. In the same vein, *any continuous real function on* $|z| = 1$ *extends as its real part to a continuous complex function on* $|z| \leq 1$, *differentiable on* $|z| < 1$:- Let $P(x, y)$ denote the steady state temperature at any point $z = x + iy$ of the disk with the given boundary values, then $u(x, y) = P(x, y) + i[\int_{(x_0,y_0)}^{(x,y)} (\frac{\partial P}{\partial x} dy - \frac{\partial P}{\partial y} dx) + C]$, page 10, gives all such extensions. The line integral is uniquely defined because its integrand is a closed 1-form, i.e., $\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = 0$, which steady state temperature obeys; and if $Q(x, y)$ denotes the imaginary part of $u(x, y)$, $\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}$, $\frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}$ on $|z| < 1$, which imply complex differentiability. $\square$

Using complex line integrals Cauchy (1825)–Chapter II–uncovered a magical fact:–*the existence of* $f'(z)$ *on an open domain of* $\mathbb{C}$ *implies the existence of all derivatives* $f^{(n)}(z)$ and *the developability of* $f(z)$ *in complex power series*[5] *near each* z_0, i.e., the series from Fourier's treatise (1822) on heat conduction, if we replace $z - z_0$ by $re^{i\theta} = r\cos\theta + ir\sin\theta$. $\square$

This power series developability of a complex differentiable function enables its smallest *germ* to spread out, almost on its own, to the maximal connected or *Riemann surface* that is its natural domain. These can even go beyond $\mathbb{C}$ itself, for example, all closed orientable surfaces occur thus. On the other hand 'any curve whatever of the plane ... under certain hypothesis of a general character concerning the curve'–Goursat II, § 87 of Ch. IV–is the *natural boundary* of a complex differentiable function. This brings us to the genesis of *Weierstrass's examples, the series* $\sum b^n z^{a^n}$ of vol. II, § 88:- for the values of a and b considered in vol.I, §200, it converges for $|z| \leq 1$, diverges for $|z| > 1$, and on $|z| = 1$ its real part coincides with the trigonometric series there. This converged uniformly to a continuous function of θ not differentiable on a dense subset–this is quite easy, and nowhere differentiability for *all* these values of a and b was finally shown by Hardy (1916)–so $|z| = 1$ is the natural boundary. $\square$

[5] Around 1966 I had worked out *a treatment of these results that avoids integration*; not so successful was a theory of *differentiabilty over the quaternions.*

5. Thus in particular that, *an infinite sum of sinusoidal waves, each of twice the frequency but half the amplitude of the previous, is continuous but nowhere differentiable,* was proved long after Weierstrass; but in the interim many had scooped my 1966 'discovery' that, *this is easy if we use a triangular wave*; the earliest apparently Takagi (1901)[6], and this function is nowadays named most often after him. Maybe, using Fourier series of piecewise linear waves, we can deduce from such results their sinusoidal counterparts?

Re-reading again after all these years that easy argument in Riesz and Sz.-Nagy the 'discovery' I made this time around was: *there are in p-adics analogous continuous functions that are nowhere differentiable.* Surprisingly in all the books on analysis over these numbers that I browsed–then there was none, now there are many–there was no mention of this fact at all. So I decided to write up this exciting story from scratch, i.e., starting from how this curious arithmetic in my opinion arose naturally from the musings of an imaginative child in primary school[7]; besides I have endeavoured to show–those who have read PG&R will know why this is important to me–that, *there is definitely more than a mere analogy between the real example and its adic version.*

Singh (1935) *is king*[8] but this nice early survey I recalled only while browsing Jarnicki and Pflug (2015), from which I also learnt that I'd been scooped again: Rychlik (1920) *gave another example of a continuous function over the p-adics which is nowhere differentiable*, but he did not tie it to a real example[9] and nor have I looked into this problem so far.

Kowalewski's *Über Bolzanos nichtdifferenzierbare stetige Funktion* (1923) is a joy to read; here's my translation of this beautiful paper, also it led to an overdue first paper in German; for more on Bolzano and how his banned *Schriften* came out a century later see also the MacTutor website. Mulling further I was led on to that cute variant of Bolzano's function, tied to *a non-controversial*[10] *fable about the tortoise and the hare,* in which the two are good friends, and always tie their races on the prescribed track, even though the rabbit is running always at twice the speed of his pal!

11/11/24 (to be continued)

[6]Alas, I couldn't find this Japanese paper, only a later translation.

[7]Browsing the relevant first two chapters of Hensel (1908) I got the impression this primary school arithmetic was known in his neck of the woods by then; even so striking is the use made of this curiosity in algebraic number theory; also some paragraphs in this book were about the only ultrametric calculus around for many decades.

[8]Of these scholarly lectures–delivered in Lucknow!–I'd learnt in the late seventies when one day I stumbled on *Squaring the circle and other monographs* by E.W. Hobson et al., including A.N. Singh's *The theory and construction of non-differentiable functions.*

[9]A 1923 translation of this Czech paper has however an end note on the recent discovery of Bolzano's example. As Hykšová, *Life and Work of Karel Rychlík* (2001) points out, though his rôle in pushing further Hensel's ideas in number theory is well-known, this early one-off paper on p-adic continuous functions came way before the topics of p-adic analysis covered in books now; apparently it was just forgotten in the folds of time.

[10]Let me recall that in all versions current among rabbits it is the tortoise who is the villain, for example in *Tricky Tortoise* their forefather is enticed into eating too many carrots before the race, so naturally he feels sleepy midway, etc.; also see *A Turtle's Tale* (2022) though its focus is a much more serious and non-mathematical issue.