HI

① Dsequs. $f(z) = \pm f(1-z)$ .. RH . . .

② Shifted complexes → calculations, inequalities
local cohomology . . VKO . . .
and semi simpl. ??
Comb. shifting here.
] purely combinatorial parts.

③ Sq. root inequality & 4KO ineq. | $\subset$ $b^K e^{t_i}$

④ Compressed KK & Mac. thms.

⑤ $\mathbb{Z}_2$-shifted complexes. (defn ??) → $\neq \overset{VKO}{0}$ iff. $K \supset T$.

⑥ Shifting | Ext. / Sym. } ↕ Joni.

⑦ $\mathbb{Z}_2$-shifting ? ✓

⑧ de Rham theory Internal $\mathbb{Z}_2$-shifting

h ⑨ Minimal models

mli

Chapter V. Heawood Inequalities

§1. A functional equation.

We will use notations f_K and $f_K(z)$ to denote the *face vector* and *face polynomial* of K respectively. These enumerate the number $f_i(K)$ of i-faces of K.

DEHN-SOMMERVILLE EQUATION. *If the links $Lk_K(\sigma)$ of a simplicial complex K have the Euler characteristics of $(dimK - |\sigma|)$-dimensional spheres, then the face polynomial*

$$f_K(z) = 1 - f_0(K) \cdot z + f_1(K) \cdot z^2 - \cdots,$$

satisfies the equation

$$f_K(z) = (-1)^{dim(K)+1} \cdot f_K(1-z).$$

Proof. By comparing coefficients of z^i on the two sides, we see that the desired functional equation is equivalent to the equations,

$$(-1)^i f_{i-1}(K) = (-1)^{dimK+1}[f_{i-1}(K) - \binom{i+1}{i} f_i(K) + \cdots],$$

which can be rewritten as

$$(1 + (-1)^{dimK-i}) \cdot f_{i-1}(K) = \binom{i+1}{i} f_i(K) - \binom{i+2}{i} f_{i+1}(K) + \cdots.$$

This follows by adding the $f_{i-1}(K)$ equations

$$\chi(Lk_K\sigma^{i-1}) = 1 + (-1)^{dimK-i} = f_0(Lk_K\sigma^{i-1}) - f_1(Lk_K\sigma^{i-1}) + \cdots,$$

since each cardinality $i+j$ simplex of K occurs as a $j-1$-simplex in the link of each of its $\binom{i+j}{i}$ $(i-1)$-dimensional faces. **q.e.d.**

Note that, by virtue of this functional equation, the zeros of the polynomial $f_K(z)$ are situated symmetrically with respect to the real line and the line $Re(z) = \frac{1}{2}$.

Replacing z by $\frac{z}{z-1}$ in $f_K(z)$, i.e. by subjecting $f_K(z)$ to a conformal transformation which replaces its line of symmetry $Re(z) = \frac{1}{2}$ by the unit circle $z = 1$, we get a rational function whose denominator can be cleared by multiplying with the $(dim(K)+1)$th power of $(1-z)$. The resulting polynomial will be denoted $h_K(z)$, so

$$(1-z)^{dim(K)+1} \cdot f_K(\frac{z}{z-1}) = h_K(z) = h_0 + h_1 \cdot z + h_2 \cdot z^2 + \cdots .$$

Note that, for this new polynomial, the functional equation reads

$$h_K(z) = z^{dimK+1} \cdot h_K(\frac{1}{z}).$$

Of interest also to us will be the polynomial $g_K(z)$ whose coefficients are the differences of the successive coefficients of $h_K(z)$, i.e. the polynomial

$$g_K(z) = (1-z) \cdot h_K(z).$$

The apparently arbitrary numbers h_i introduced above turn out to have many natural interpretations, some of which we are in a position to give in this section itself. (See also §§5 and 11.)

§2. Some "Riemann hypotheses".

It is natural to ask for a characterization of those K for which the zeros of $f_K(z)$ lie on the lines of symmetry themselves.

Though this looks hard, it is easy enough to check the following.

THEOREM 1. *Any manifold X admits triangulations K for which the non real zeros of the polynomial*

$$\frac{\chi(K)}{2} - f_0(K) \cdot z + f_1(K) \cdot z^2 - \cdots$$

lie on the line $Re(z) = \frac{1}{2}$.

Proof. We will only use the fact that if $X \cong |K|$, then the link of any nonempty simplex σ has the Euler characteristic of a $(dimX - |\sigma|)$-dimensional sphere. [An X satisfying this condition is sometimes called an *Euler manifold.*]

Using this fact it follows again (as in §1) that our polynomial remains the same up to a sign (depending upon the parity of $dimX$) if z is replaced by $1 - z$.

So, for *any* triangulation of X, the roots of this polynomial are situated symmetrically with respect to the real axis and the line $Re(z) = \frac{1}{2}$.

Furthermore these polynomials take real values on the real axis, and, depending upon the parity of $dimX$, either purely imaginary, or else real, values on the line $Re(z) = \frac{1}{2}$.

Deriving a top m-dimensional simplex q times adds to such a polynomial the terms

$$-qz(1-z)^{m+1} - (-1)^{m+1}qz^{m+1}(1-z).$$

It is easy to see that this last polynomial (whose values have the same behaviour with respect to the two lines) has distinct roots, of which two, *viz.* 0 and 1, lie on the real axis, and the remaining satisfy $|\frac{z}{1-z}| = 1$, i.e. lie on the line $Re(z) = \frac{1}{2}$.

But, for q big, the complete polynomial of such a derived triangulation is only a small perturbation of the above terms. So, using continuity and the fact that its values on the two lines are constrained as above, it too will have distinct roots, two on the real axis, and the remaining on the line $Re(z) = \frac{1}{2}$. **q.e.d.**

REMARK. (1) *On the other hand, manifolds also have many triangulations for which the above "Riemann hypothesis" does not hold.*

To see this take for example any neighbourly triangulation of a 3-manifold with N vertices, and so $\binom{N}{2}$ edges, with $N > 6$. The verification can be done by making the substitution $z = \frac{1}{2} + z'$. The resulting polynomial has no linear and degree 3 terms, i.e. is a quadratic in $(z')^2$. One verifies quickly that the roots of this quadratic are not real, as they should be for the RH to hold.

Before putting forward a more interesting Riemann hypothesis for simplicial complexes, we examine the combinatorics involved in the proof of a well known topological result of Lefschetz.

A sequence $\sigma_1, \sigma_2, \ldots, \sigma_r$ of j-simplices of K', a subdivision[1] of K, will be called an *orbit* of a simplicial map $\nu : K' \to K$, if

$$\sigma_1 \subseteq \nu(\sigma_r), \sigma_2 \subseteq \nu(\sigma_1), \cdots, \sigma_r \subseteq \nu(\sigma_{r-1}),$$

but no shorter subsequence satisfies these conditions.

LEFSCHETZ FIXED POINT FORMULA. *Let N_j^r denote the number of such orbits, of length r, of j-simplices of K', under a simplicial map $\nu : K' \to K$. Also let $\nu_* : H_j(X;\mathbb{Q}) \to H_j(X;\mathbb{Q})$, $X = |K'| = |K|$, be the induced map in rational homology.*

[1]For application to the Riemann hypothesis below we in fact only require the case $K' = K$

Then

$$\sum_j (-1)^j N_j^r = \sum_j (-1)^j Tr_j(\nu_*)^r,$$

and so also

$$exp \sum_r \sum_j (-1)^j N_j^r \frac{t^r}{r} = \prod_j (det(Id - t\nu_*))^{(-1)^j}.$$

Proof. ...
... **q.e.d.**

REMARK. (2) The proof of the *topological* fixed point formula continues as follows:

If ν is a simplicial approximation of a "nice" continuous map $\nu : X \to X$, then $\sum_j (-1)^j N_j^r$ *interprets as the number* N_r *of fixed points of* ν^r, each counted as many times as the *local degree* , i.e. the alternating sum of the traces of the maps ν_*^r induced in the homology of its link.

(3) Thus for the "zeta function" of ν, i.e. the following power series which enumerates the number of orbits of ν, one has the formula

$$exp \sum_r N_r \frac{t^r}{r} = \prod_j (det_j(Id - t\nu_*))^{(-1)^j}.$$

For any simplicial complex K we will now consider a *zeta function* which enumerates the alternating sum of the pairwise disjoint r-tuples of simplices of K, for all $r \in \mathbb{N}$, and then use the above Lefschetz formula to display it as a rational function.

After this, we will consider the relationship of this zeta function to other zeta functions, and pose the questions regarding its functional equation and Riemann hypothesis.

It seems likely that this last conjecture is closely connected to some well known chromatic problems. To explore this we also take a look at the "critical problem" formulation of such questions.

§3. **Shifted complexes.**The vertices of all our simplicial sets will always be taken from a fixed universal set $\mathcal{V}$.

A simplicial set K is said to be *shifted* with respect to a given total order $\leq$ on $vert(K)$ iff it is closed in the induced product partial order $\leq$ on the set of subsets of $vert(K)$.

Alternatively the above definition can be rephrased in terms of the operations

$$K \rightsquigarrow \Delta_{uv}(K).$$

Here $u, v \in \mathcal{V}$, and the operation is simply to replace in any simplex, whenever feasible, the vertex v by the vertex u. (Thus this operation comes equipped with the bijection $\Delta_{uv}^K : K \to \Delta_{uv}(K)$, which maps a simplex σ to σ, unless $v \in \sigma$, $u \notin \sigma$, and $(\sigma \setminus v) \cup u \notin K$, when $\Delta_{uv}^K \sigma = (\sigma \setminus v) \cup u$.)

Obviously K is shifted with respect to a given total order on $vert(K)$ if and only if it is invariant under all operations Δ_{uv} with $u, v \in vert(K)$ and $u < v$.

It is clear that by performing enough of these operations, any given K can be changed to another which is shifted. Despite their *ad hoc* nature these *combinatorial shifting operations* do have some interesting properties.

THEOREM 1. *The operation $K \rightsquigarrow \Delta_{uv}(K)$ on simplicial sets, preserves their face vectors, and does not increase those of their deleted joins. Further, it commutes with inclusions and shrinks shadows, i.e $L \subset K \implies \Delta_{uv}(L) \subset \Delta_{uv}(K)$ and $\partial\Delta_{uv}(K) \subset \Delta_{uv}(\partial K)$, and so maps simplicial complexes to simplicial complexes.*

Here the *shadow* ∂K of K consists of all simplices θ which are codimension one faces of simplices of K.

Proof. That $|K| = |\Delta_{uv} K|$ is clear.

To see $|K_*| \geq |(\Delta_{uv} K)_*|$ note that the bijection of pairs $(\sigma, \theta) \mapsto (\Delta_{uv}^K \sigma, \Delta_{uv}^K \theta)$ takes a non-disjoint pair to a disjoint one iff $\sigma \cap \theta = v$ and v can be replaced by u in precisely one of the simplices. If this be σ, then the disjoint pair $(\sigma, (\theta \setminus v) \cup u)$ of K goes under the bijection to a non-disjoint pair of $\Delta_{uv}(K)$. Thus the number of non-disjoint pairs becoming disjoint under the bijection is no more than the number of disjoint pairs becoming non-disjoint.

If in a simplex σ of $L \subset K$, one can replace v by u, but *not* when σ is considered as a member of K, then $\Delta_{uv}^L \sigma = (\sigma \setminus v) \cup u$ must already be in K, and thus also in $\Delta_{uv}(K)$.

The verification regarding shadows is also straightforward (but a little messy).

Finally, if K is a simplicial complex, i.e. if $\partial K \subset K$, then so is $\Delta_{uv}(K)$, because these 2 properties give $\partial(\Delta_{uv}(K)) \subset \Delta_{uv}(K)$. **q.e.d.**

REMARK. The Kuratowski graph $K = \sigma_0^2 \cdot \sigma_0^2$ gives an example where one can have strict inequality $|K_*| > |(\Delta_{uv} K)_*|$.

In fact (see *fig.1*) there are only 2 connected shifted graphs having 6 vertices and 9 edges. And, there is just one more, *viz.*the disjoint union of a point and $\sigma_1^4 \setminus \{an\ edge\}$, which is disconnected. In all three cases the deleted join is smaller than that of $\sigma_0^2 \cdot \sigma_0^2$. So

(0) *one cannot define a shifting operation on simplicial complexes, which will preserve their face vectors, as well as those of their deleted joins.*

We will be interested in other analogous problems of deciding whether there exists a shifting operation preserving some given properties, and, in case there is, in deducing some combinatorial conclusions from its existence. For these purposes we now list some properties of shifted complexes (the statements of various parts of the theorem being interspersed with their proofs).

THEOREM 2. SOME COMPUTATIONS FOR SHIFTED COMPLEXES. *Let K^n be a shifted n-complex. Then*

(a)$H_i(K) \cong \mathbb{Z}^{\beta_i(K)}$, *where $\beta_i(K)$ is the number of maximal i-simplices of K not containing the first vertex.*

Proof. This follows because, being shifted, the simplicial complex K is the union of a cone over the first vertex, and these $\sum_i \beta_i$ maximal simplices not containing the first vertex.

This shows in fact that

(b) *K has the homotopy type of a bouquet of $\sum_i \beta_i$ spheres, of which β_i are i-dimensional.*

(c) *K^n is* Cohen-Macaulay , *i.e. links of all $\sigma \in K^n$ have trivial homologies in dimensions less than $n - |\sigma|$, iff K^n is homogenously n-dimensional.*

A CM complex K^n can not have a maximal σ^i with $i < n$, for then $H_{-1}(Lk_K\sigma^i) = H_{-1}(\emptyset) \neq 0$ and $-1 < n - i + 1$.

Conversely, for each $\sigma \in K$, $Lk_K\sigma$ being both shifted and homogenously $(n-|\sigma|)$-dimensional, we see by applying **(a)** that $H_i(Lk_K\sigma) = 0$ $\forall i < n - |\sigma|$.

If the above links condition is required only for *nonempty* σ, then K^n is called *Buchsbaum.*

(d) *K^n is Buchsbaum only if it is Cohen-Macaulay.*

This follows by **(c)** because the first part of its proof is true even now and shows that K^n is homogenously n-dimensional.

(e) *K^n is m-Leray , i.e. links of all $\sigma \in K$ have trivial homologies in dimensions $\geq m$, iff every maximal simplex with $m+k$ vertices contains the first k vertices.*

Since $Lk_K\sigma$ is shifted **(a)** shows that $H_i(Lk_K\sigma) \neq 0$ for some $i \geq m$ iff K has a maximal simplex θ of cardinality $m+k = i+1+|\sigma|$ containing σ, but not the least vertex v outside σ. This v is amongst the first $|\sigma|+1$, and so amongst the first k, vertices of K.

Conversely, if ξ consists of all vertices of θ less than v, then **(a)** shows $H_i(Lk_K\xi) \neq 0$ for some $i \geq m$.

(f) *$\mathfrak{o}_{m+1}(K_*) \in H^{m+1}(K_*; \mathbb{Z}_2)$ is nonzero, and thus K^n does not embed in $\mathbb{R}^m$, whenever $f_t(K^n) \geq (m-t+2) \cdot f_{t-1}(K^n)$ with $t \leq n \leq m \leq 2t$.*

The number of t-simplices of K^n having a specified first vertex being less than $f_{t-1}(K)$, we see that the given inequality can hold only if there is a $\phi^t \in K^n$ which does not contain any of the first $m-t+2$ vertices, and so has biggest vertex not less than the $(m+3)$rd. The sub complex of K^n *dominated* by ϕ^t, i.e. determined by all t-simplices $\leq \phi^t$, contains a $\sigma^{2t-m-1}_{2t-m-1} \cdot \theta^{2(m-t)+2}_{m-t}$: the second part being determined by the $2(m-t)+3$ vertices ending with the $(m+3)$rd, and the first part by the preceding $(m+3)-(2(m-t)+3) = 2t-m$ vertices. But this has $(m+1)$th van Kampen obstruction nonzero by Th. ... of Ch. II.

The kind of obstructions met in the last argument suggested the next definition and result.

(g) *K^n is a* $\mathbb{Z}_2$-Cohen-Macaulay complex , *i.e. the deleted join of the link of any $\sigma \in K^n$ has trivial $\mathbb{Z}_2$-homology in dimensions less than $n-|\sigma|$, iff K^n is homogenously n-dimensional and contains no $\sigma^i_i \cdot \theta^j_{2j-1}$ with $i+2j=n$.*

Since $(\emptyset)_* = \emptyset$ the homogeneity follows using the same argument as in **(c)**. And, if $\sigma^i_i \cdot \theta^j_{2j-1} \subset K$, then $Lk_K\sigma$ contains a top $(2j-1)$-dimensional sphere $(\theta^2_{k-1}k)_*$.

Conversely, each $Lk_K\sigma$ being shifted, by **(h)** below the homology of its deleted join is given by $H_{2i+1}(K_*) \cong \mathbb{Z}^{\beta_{t,t}}$, where $\beta_{t,t}$ is the number of maximal, i.e. i-simplices, which miss the first vertex outside σ. The presence of such a pair is equivalent to a $\theta^2_{j-1}j$ in the link.

(h) *$H_{i,j}(K_*) \cong \mathbb{Z}^{\beta_{i,j}(K)}$, where $\beta_{i,j}(K)$ is the number of disjoint max-*

imal pairs of i- and j-simplices of K not containing the first vertex.

For the proof note that ...

(j) $o \in H^{2n+1}(K_*; \mathbb{Z}_2)$ *is nonzero iff* K^n *contains a Kuratowski n-complex* T^n. *However, such a* K^n *may not contain the irreducible Kuratowski n-complex* σ_n^{2n+2}.

The second part follows because the following shifted and non-planar graph does not contain a complete graph on 5 vertices.

For the first part note that ... ... **q.e.d.**

MORE REMARKS. By **(a)** the homology of a shifted complex has no torsion. Thus

(1) *there is no shifting operation on simplicial complexes which will preserve integral homology.*

In particular note that the defining bijections Δ_{uv}^K of the combinatorial shifting operation can not be monotone.

Simplicial homology manifolds,*resp.* spheres, give examples of Buchsbaum, *resp.* Cohen-Macaulay, complexes. Also note that

(2) *simplicial spheres are* $\mathbb{Z}_2$*-Cohen-Macaulay.*

The local conditions, ($\mathbb{Z}_2$-)Cohen-Macaulay, m-Leray, Buchsbaum,etc. are modified in the obvious way as one switches from integral, to some other coefficients. With field coefficents (but not with integers!) it is possible to define suitable shifting operations which will preserve these properties, except that by **(d)** the property Buchsbaum gets mixed with the more stringent property Cohen-Macaulay, and thus

(3) *shifting is not the right way to examine f-vectors of manifolds other than spheres.*

... ...

§4. Contractions.

We now return to combinatorial shifting, and consider its behaviour *vis-à-vis* the above homological properties.

For this purpose the map $\Delta_{uv}^K : K \to \Delta_{uv}(K)$ is not useful because it is monotone *only* in the trivial cases, i.e. only if $u \notin vert(K)$ or $\Delta_{uv}(K) = K$, when it is merely the identity map or the renaming of v as u.

So we will consider the map $\Gamma_{uv}^K : K \to \Delta_{uv}(K)$, which equals Δ_{uv}^K if $u \notin vert(K)$ or $\Delta_{uv}(K) = K$, and otherwise is the simplicial map which preserves all vertices other than v, and maps v to u. Thus its image $\Gamma_{uv}(K)$, when less than $\Delta_{uv}(K)$, is the complement in $\Delta_{uv}(K)$ of the vertex v, as well as the simplicial set obtained from K by identifying the vertex v with u.

Thus, for each $u, v \in \mathcal{V}$, we have defined a *combinatorial contraction operation*

$$K \rightsquigarrow \Gamma_{uv}(K),$$

which comes equipped with a simplicial surjection $\Gamma_{uv}^K : K \to \Gamma_{uv}(K)$.

In case the vertices are totally ordered, by performing enough of these operations Γ_{uv}, with $u, v \in vert(K)$ and $u < v$, such a K would eventually change to another which is shifted. Note that if K were already shifted it would remain unchanged. Thus, though this shifting operation may, and usually does, decrease the face vector, it too is a non trivial projection onto the shifted simplicial sets.

A Cohen-Macaulay complex K^n may admit plenty of operations Δ_{uv} resulting in an impure, and so non-Cohen-Macaulay, complex $\Delta_{uv}(K^n)$. However it seems that purity is the only obstruction.

'THEOREM' 1. *If K^n is Cohen-Macaulay, and $\Gamma_{uv}(K)$ is homogenously n-dimensional, then $\Gamma_{uv}(K)$ is also Cohen-Macaulay.*

Proof. We have to show

$$H_i(Lk_\Gamma \sigma) = 0 \ for \ i < n - |\sigma|,$$

for all $\sigma \in \Gamma = \Gamma_{uv}(K)$.

Since Γ^n is pure, this is so for the maximal simplices σ of Γ.

So assume inductively that σ is not maximal, and that the above property has already been verified for simplices bigger than σ.

Using this inductive hypothesis we now verify that, *for any $i < n - |\sigma|$, a homology class of $H_i(Lk_\Gamma \sigma)$ can be represented by an i-cycle z which does not contain any given vertex w of K*:

Indeed, if an i-cycle z' of $Lk_\Gamma \sigma$ contains w, then 'dividing out' by w we can write $z' = wq + r$ where, since r does not contain w, we will have $\partial q = 0$ and $q + \partial r = 0$. Thus q is an $(i-1)$-cycle of $Lk_\Gamma \theta$ where $\theta = w \cdot \sigma$ is bigger than σ. So $q = \partial c$ for some i-chain c of $Lk_\Gamma \theta \subset Lk_\Gamma \sigma$. Now consider $z = c + r$. Since $\partial c + \partial r = q + \partial r = 0$, z too is an i-cycle of $Lk_\Gamma \sigma$. Further $z - z' = c - wq = \partial(wc)$. So z is homologous to z'.

Case 1, $u \notin \sigma$. By above, it would be enough to show that i-cycles z of $Lk_\Gamma \sigma$,*not containing* u, bound in $Lk_\Gamma \sigma$. Now $\sigma \cdot z$ is also contained in $Lk_K \sigma$ where it bounds some $(i+1)$-chain c. The simplicial image $\Gamma(c)$ is contained in $Lk_\Gamma \sigma$ and bounds z.

Case 2, $u \in \sigma$. Consider the *subcase* $\sigma \notin K$. So $\sigma = u\xi \notin K$ but $v\xi \in K$, likewise any bigger simplex of Γ, $u\theta \supseteq u\xi \notin K$ but $v\theta \in K$. Also note $u \notin Lk_K v\xi$, otherwise $u\xi \in K$. Thus $Lk_\Gamma u\xi = Lk_K v\xi$ and result follows.

But the *subcase* $\sigma \in K$ eludes us: Now for some of Γ's simplices of type $u\theta \subseteq u\xi = \sigma$ we may have $u\theta \in K$, and thus may or may not $v\theta \in K$, but for others $\mathring{u}\theta \notin K$, and so surely $v\theta \in K$: Possibly an argument can still be pushed through combining information from more than one link

... ...

Remark. We will show later that *if* K *is Cohen-Macaulay and* $\{u, v\}$ *is an edge of* K*, then* $\Delta_{uv}(K)$ *is also Cohen-Macaulay.*

It seemed that this could be proved by first showing merely that $\Delta_{uv}(K)$ is pure, and then applying the above 'result' which apparently holds even for Δ_{uv} instead of Γ_{uv}. But both parts of this strategy run into heavy weather.

The proof given later will be by constructing (using the fact that K is CM) a chain map analogous to (but more complicated than) the one used in Theorem 4 below.

We define a binary relation $\mathcal{K}$ on $vert(K)$ as follows: $u\mathcal{K}v$ iff for any simplex of K which contains u but not v, the replacement of a vertex other than u by v results in a simplex of K.

The symmetric relation generated by $\mathcal{K}$ will be called *matroidal adjacency*, and the equivalence classes of $vert(K)$, under the equivalence relation generated by $\mathcal{K}$, will be called the *matroidal components* of K.

So any K^0 is *matroidally connected*, i.e. has only one matroidal component, while a graph K^1 is matroidally connected iff all its ordinary components, with the exception of just one, are only isolated vertices. For $n \geq 2$, the matroidal connectivity of K^n is obviously far stronger than connectivity.[2]

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[2]We will later give another operation having analogous results under much less restrictive conditions.

THEOREM 2. *If u and v are matroidally adjacent in K^n, and $\mathfrak{o}_{2n+1}(K_*) = 0$, then also $\mathfrak{o}_{2n+1}((\Gamma_{uv}(K)_*) = 0$.*

Proof. We construct a chain map from the chains of $\Gamma = \Gamma_{uv}$ to the chains of K', the stellar subdivision of K at the barycentre t of the edge $\{u, v\}$.

For this purpose we map any simplex not containing the vertex u of Γ to itself. If a simplex $\sigma = u \cdot \theta$ containing u is non-maximal, then $u \cdot \theta$ or $v \cdot \theta$ (or both) are non-maximal simplices of K, and so, by virtue of uKv or vKu, the simplex $\{u, v\} \cdot \theta$ also belongs to K. We now map σ to the simplex $t \cdot \theta$ of K'. And, finally if σ is maximal, then to one of the chains

$$u \cdot \theta + u \cdot t \cdot \partial\theta \text{ or } v \cdot \theta + v \cdot t \cdot \partial\theta,$$

which makes sense since at least one of these is in K'.

The verification that this is indeed a chain map is straighforward, so there is also a dual cochain map $C(K') \to C(\Gamma)$.

Since disjointness of simplices is preserved, there is an induced $\mathbb{Z}_2$-chain map from the chains of Γ_* to those of $(K')_*$. And, thus there is a dual *$\mathbb{Z}_2$-cochain map $C((K)'_*) \to C(\Gamma_*)$ which maps 1 to 1.*

The above construction is valid with any coefficients. Taking *mod* 2 coefficients the result now follows by Theorem .. of Chapter II. **q.e.d.**

REMARKS. (1) *Even under above hypothesis $\Delta_{uv}(K)$ need not have van Kampen obstruction zero.*

For example the following 6 vertex planar graph K, after the operation Δ_{12}, contains the Kuratowski graph $\{2, 5, 6\} \cdot \{1, 3, 4\}$, which of course does not lie in the contraction $\Gamma_{12}(K)$.

(2) *Even with u matroidally adjacent to v, neither Γ_{uv} or Δ_{uv} need preserve non-planarity of a graph.*

For example contract any edge of the Kuratowski graph $\sigma_0^2 \cdot \sigma_0^2$ to see this.

(3) *It is conceivable that contractions of the kind considered in above result may preserve embeddability in any 2n-manifold M^{2n}.*

We will consider this geometric problem later, as well as the connections with the Robertson-Seymour theory.

The next proof is in fact the definition of a canonical chain isomorphism.

THEOREM 3. *If u and v are matroidally adjacent in K, then the integral homology of K is isomorphic to that of $\Delta_{uv}(K)$.*

Proof. There is a homomorphism of chain groups, $C(\Delta) \rightarrow C(K)$, $\Delta = \Delta_{uv}(K), u, v \in$ $vert(K)$, defined thus:

If a maximal simplex σ of Δ contains u but not v, say $\sigma = u \cdot \theta$, then map it to itself if $u \cdot \theta \in K$, otherwise to $v \cdot \theta + u \cdot v \cdot \partial\theta$. All other simplices are mapped to themselves.

Obviously this homomorphism is one-one.

It is onto because if maximal $u \cdot \theta \in \Gamma$ is such that $u \cdot \theta \notin K$, then $v \cdot \theta$ is the image of the chain $u \cdot \theta - u \cdot v \cdot \partial\theta$.

Lastly it is a chain map because, in the above case, the boundary of $v \cdot \theta + u \cdot v \cdot \partial\theta$ is $\theta - u \cdot \partial\theta$, i.e. the image of $\partial(u \cdot \theta) = \theta - u \cdot \partial\theta$. **q.e.d.**

REMARKS. (4) *The above result shows that a sequence of operations Δ_{uv}, $u < v$ and matroidally adjacent, can result in a shifted complex only under severe homological restrictions on K.*

This follows because the homology of a shifted complex has no torsion etc.

(5) *Note that, for u and v matroidally adjacent, we have constructed a $\mathbb{Z}_2$-cochain map $C(K_*) \rightarrow C(\Gamma_*)$ taking* 1 *to* 1. (So the use of K' in the proof of Theorem 2, was not essential.)

This follows because, restricted to the chains of Γ, the map preserves disjointness. (Note that the present chain map is a composition of a map of the kind considered for the proof of Theorem 2, and a simplicial inverse of the chain subdivision map $C(K) \rightarrow C(K')$.)

(6) *Unless $\Delta_{uv}(K) = K$, the bijection $\Delta^K_{uv} : K \rightarrow \Delta_{uv}(K)$ is not monotone*, even under the above hypothesis, and so can't be used for the above result: some maximal simplex does move under this bijection, while its proper faces do not.

We now analyse the relation $\mathcal{K}$ more thoroughly.

THEOREM 4. PROPERTIES OF MATROIDAL ADJACENCY.

(a) *If K is shifted, then it is matroidally connected.*

Proof. Assume inductively that the first t vertices are in the same equivalence class. Then the $(t+1)$th vertex u is either isolated, when

$u\mathcal{K}v\ \forall v$, or else there exist vertices $v < u$ such that $\{u, v\} \in K$. One has $u\mathcal{K}v$ for the least such vertex.

A simplex $\sigma \subset vert(K)$ is called a *circuit* of K if $\sigma \notin K$, but all proper faces of σ are in K.

(b) *Let K be a neighbourly matroid. Then u, v can lie in the same circuit of K only if $u\mathcal{K}v$ and $v\mathcal{K}u$.*

Recall that K is called a *matroid* if any simplex can be augmented to the size of a bigger one by adding vertices from the latter.

Let $\theta \cdot t \cdot u \in K$ be a maximal simplex not containing v. From the hypothesis it follows that $\theta = \phi \cdot \xi$ with $\phi \cdot u \cdot v$ a circuit of K. Now augment $\phi \cdot v$ to the size of $\theta \cdot t \cdot u$ by adding vertices from the latter. Since this process can not use u, the simplex we'll get is $\theta \cdot t \cdot v$.

(c) *If $u\mathcal{K}v$, and K is matroidally connected, then so is $\Gamma_{uv}(K)$.*

If $a, b \in K$ are different from u and v and $a\mathcal{K}b$, then we have $a\mathcal{G}b$. (Here $G = \Gamma_{uv}(K)$.) Same holds if $b = u$. The least non-trivial case is $a = u$ when, for the same conclusion, we also use the given hypothesis $v\mathcal{K}u$. **q.e.d.**

REMARKS. (7) *In a shifted complex there may be vertices u, v which are not matroidally adjacent.*

For example $3, 4$ in the following

(8) *Even if all circuits of a matroid K have cardinality ≥ 3 one can have $u\mathcal{K}v$ and $v\mathcal{K}u$, and yet u, v may not lie in a circuit of K.*

For example if u lies in the first, and v in the second, factor of a $\sigma_0^0 \cdot \sigma_1^2$, then this is so.

(9) *For matroids it is known that the binary relation* 'in the same circuit' *is transitive, and so an equivalence relation.*

By **(b)** and (8) it follows that for neighbourly matroids this is a (possibly strictly) smaller equivalence relation than that generated by $\mathcal{K}$. So perhaps it was not wise to call this last relation 'matroidal connectivity' because, for the case of matroids, this terminology is traditionally used for the former.

THEOREM 5. *If σ_n^{2n+2} occurs in the end result of a sequence of shiftings employing the operations Δ_{uv}^K with $u\mathcal{K}v$ always, then it must also occur in the end result of some (possibly other) sequence of contractions employing the operations Γ_{uv}^K with $u\mathcal{K}v$ always.*

The same assertion is *not* true for the reducible Kuratowski complexes: see the example given in Remark (2) above.

Proof. ...... **q.e.d.**

It seems that the above "lifting" result in fact holds for any join-irreducible matroid in place of σ_n^{2n+2}.

§5. Semi-simplicial (co)chains.

The standard oriented chain complex is inconveniently small for some algebraical-topological constructions. A natural combinatorial enlargement of it is obtained by considering those singular simplices, i.e. continuous maps

$$\xi : \{0, 1, \ldots, q\} \to K,$$

which are simplicial . These will be called the (non commutative) *semi-simplices* of K, and finite linear combinations of these with specified coefficients, will form the chain groups $\mathcal{L}_q(K)$.

We now equip $\mathcal{L}(K)$, the direct sum of these chain groups, with the *boundary operator* $\partial : \mathcal{L}_q(K) \to \mathcal{L}_{q-1}(K)$ inherited from the singular chain complex, i.e. defined by

$$\partial\xi = \sum_r (-1)^r \xi_r,$$

where the codimension-one *faces* ξ_r of ξ are, as usual, the $(q-1)$-semi-simplices

$$\{0, 1, \ldots, q-1\} \xrightarrow{\cong} \{0, 1, \ldots, \hat{r}, \ldots, q\} \xrightarrow{\xi} K.$$

One has $\partial \circ \partial = 0$ not only for the $\mathcal{L}(K)$ of a simplicial complex K, but also for an $\mathcal{L} = \mathcal{L}(\mathcal{B})$ spanned by a *semi-simplicial complex* $\mathcal{B}$, i.e. a set of semi-simplices which also contains all their faces. (Thus $\mathcal{L}(K) = \mathcal{L}(\mathcal{B}(K))$, where $\mathcal{B}(K)$ consists of all semi-simplices of K.) So one can speak of *semi-simplicial homology* .

The dual definitions of semi-simplicial cochains, and their coboundary operators δ, and thus of *semi-simplicial cohomology*, are exactly what one would expect, and so have been omitted.

We will also be interested in *commutative q-semi-simplices* $[\xi]$, i.e. totality of q-semi-simplices related via permutations of $\{0, 1, \ldots, q\}$. For example, for any simplicial complex K, these determine a quotient complex $B(K)$ of $\mathcal{B}(K)$, and thus a quotient chain complex $L(K)$ of $\mathcal{L}(K)$.

Dually there is the sub cochain complex $L^*(K) \subset \mathcal{L}^*(K)$ of commutative cochains.

The commutative chain complex $L(K)$ will be used usually when $vert(K)$ comes equipped with a specified total order (or at least a partial order which restricts to a total order on each simplex of K). Considering this, it is natural to identify this quotient complex $L(K)$ of $\mathcal{L}(K)$ with the subcomplex of $\mathcal{L}(K)$ determined by all ξ's which are *order-preserving*.

Such an order-preserving ξ determines the *monomial* of vertices,

$$\prod_{x_i \in vert(K)} {x_i}^{q_i}, \quad \sum_i q_i = q + 1, \ \{x_i : q_i > 0\} \in K,$$

where the exponent q_i of the ith vertex x_i equals the cardinality of $\xi^{-1}(x_i)$.

And, conversely, the simplicial map $\xi : \{0, 1, \ldots, q\} \to K$ can be recovered from this monomial by mapping the ith group of q_i vertices to x_i.

Henceforth, we will identify commutative semi-simplices and monomials as above. In particular a commutative semi-simplicial complex B (e.g. the $B(K)$ of a simplicial complex K) thus identifies with a set of monomials closed under divisibility, i.e. an *order ideal* of monomials.

Using this monomial description we now define an increasing *filtration* of semi-simplicial chains: $L^{[r]}$ will be the sub chain complex determined by those monomials in which any vertex appears with multiplicity $\leq r$.

The usefulness of this filtration becomes clear from

BIER'S THEOREM. *If r is odd, then the homology of $L^{[r]}(K)$ coincides with the homology* (of the link of the empty simplex) *of K. However, if r is even, then it is* much *bigger: it is the direct sum of the homologies of* all *the links of K. In particular,*

(a) *K^n is Cohen-Macaulay iff $H_q(L^{[r]}(K)) = 0$ for all even r such that $r - 1$ does not divide $q - n$, and*

(b) *K^n is Buchsbaum iff $H_q(L^{[r]}(K), L^{[r-1]}(K)) = 0$ for all even r such that $r - 1$ does not divide $q - n$.*

Proof. We first show that the case r even of the theorem follows from the case r odd.

For this purpose denote by $L_q^{[r,\sigma]}(K)$ the subgroup of $L_q^{[r]}(K)$ determined by all those degree $q+1$ monomials for which σ is the maximal simplex of K whose rth power divides the monomial. The direct sum of these , over all q, will be denoted $L^{[r,\sigma]}(K)$. Note that $L^{[r,\emptyset]}(K) = L^{[r-1]}(K)$.

Now we come to the main point: *while taking boundaries $\partial\xi$ of monomials ξ the even powers of the vertices can be pulled out*, i.e. they behave just like constants in differentiation.

Thus, for r even, each $L^{[r,\sigma]}(K)$, $\sigma \in K$, is a sub chain complex, and so we have the direct sum decomposition of chain complexes

$$L_q^{[r]}(K) \cong \bigoplus_{\sigma \in K} L_q^{[r,\sigma]}(K).$$

Furthermore, division by the rth power of σ gives a chain isompor-phism

$$L_q^{[r,\sigma]}(K) \overset{\div\sigma^r}{\to} L_{q-r|\sigma|}^{[r-1]}(Lk_K\sigma).$$

Thus,

$$for\ r\ even,\ H_q(L^{[r]})(K) \cong \bigoplus_{\sigma \in K} H_{q-r|\sigma|}(Lk_K\sigma).$$

Summing over all q we see that $H^{[r]}(K)$ is the direct sum of the homologies of the links of all the simplices of K.

We now go to the case r odd. By once again using the above point regarding even powers behaving like constants, we see that there is a direct sum decomposition of chain complexes $L(K) = L^{[r]} \oplus L^{]r[}(K)$.

The chains a of $L^{]r[}(K)$ are *polynomials* $\sum_i a_i\alpha_i$ with each monomial α_i having at least one vertex with power $> r$. Dividing by the $(r+1)$th power of such a vertex x, we write $a = x^{r+1}b + c$, where $b = \sum_j \{a_j\beta_j : x^{r+1}\beta_j = \alpha_j\}$, and $c = \sum_k \{a_k\alpha_k : x^{r+1} \nmid \alpha_k\}$. If a is a cycle, i.e. if $x^{r+1}\partial b + \partial c = 0$, then $\partial b = 0 = \partial c$. The *total degree* (the sum of the degrees of the occuring monomials) of c being less than that of a we assume inductively that it bounds in $L^{]r[}(K)$. (This induction can commence because for any vertex x, $\partial(x^{r+2}) = x^{r+1}$.) Also, if for $x^{r+1}|\alpha_j$, the degree of the sub monomial of α_j formed by vertices less than x is n_j, then an easy computation shows that $\partial(\sum_j (-1)^{n_j} a_j x^{r+2}\beta_j) = x^{r+1}b$. So all cycles a of $L^{]r[}(K)$ bound.

We have thus shown that for r odd the inclusion map $L^{[r]}(K) \to L(K)$ induces an isomorphism in homology. But, for $r = 1$, the quotient $L^{[1]}(K)$ of $\mathcal{L}(K)$ identifies with the *oriented* chain complex $C(K)$ of

K (and dually the sub cochain complex $L^*_{[1]}(K)$ of $\mathcal{L}^*_{[1]}$ identifies with the *alternating* cochain complex $C^*(K)$). So by definition its homology coincides with that of K. So,

$$for\ r\ odd,\ H_q(L^{[r]}(K)) \cong H_q(K).$$

To see **(a)** note that if $H_i(Lk_K\sigma) = 0$ for $i \neq n - |\sigma|$, then the summand $H_{q-r|\sigma|}(Lk_K\sigma)$ of $H_q(L^{[r]}(K))$ is zero for $q - r|\sigma| \neq n - |\sigma|$, i.e. for $q - n \neq |\sigma|(r-1)$, and thus all summands are zero if $r - 1$ does not divide $q - n$.

Conversely, note that $H_i(Lk_K\sigma)$, $i < n - |\sigma|$, occurs as a summand of $H_{i+2k|\sigma|}(L^{[2k]}(K))$, for which the required condition, viz. that $2k-1$ does not divide $i + 2k|\sigma| - n \equiv i + |\sigma| - n \ mod\ 2k-1$, can be ensured by choosing k so large that $2k - 1$ is bigger than the absolute value of the negative integer $i + |\sigma| - n$.

Since,

$$for\ r\ even,\ H_q(L^{[r]}(K), L^{[r-1]}(K)) \cong \bigoplus_{\emptyset \neq \sigma \in K} H_{q-r|\sigma|}(Lk_K\sigma),$$

the same argument gives **(b)** also. **q.e.d.**

Let $L^{r,t}(K) \subset L^{[r]}(K)$ denote the subgroup determined by all monomials having *exactly* t vertices of multiplicity r. For r even, this too is a chain complex, and enters into a similar global characterization of yet another local condition.

(c) *Let r be even. Then K is m-Leray iff $H_q(L^{r,t}(K)) = 0$ for all t and q such that $q - rt \geq m$.*

This follows because $L^{r,t}_q(K) \cong \bigoplus_{|\sigma|=t} L^{[r-1]}_{q-rt}(Lk_K\sigma)$.

All order ideals B of monomials obviously do not occur as $B(K)$ of some K. However there is a useful bijective correspondence with *colored simplicial complexes* which we now investigate.

Later on we will use Bier's theorem to show that the, respectively, skewsymmetric and symmetric linear shifting operations on simplicial complexes and order ideals of monomials, defined below in §§7 and 10, preserve the properties of being Cohen-Macaulay or m-Leray.

This puts these known results of Kalai and Reisner in a *much* clearer light. For considerations related to the original proofs see §§11 and 15.

We will later also investigate the analogous filtration $\mathcal{L}^{[r]}(K)$ of $\mathcal{L}(K)$, by doing a further analysis of the argument by means of which Eilenberg and Steenrod [..] showed that the map $L^{[1]} \hookrightarrow \mathcal{L}(K)$ induces an isomorphism in homology.

§6. Compression.

To really get the shifting ball rolling it is necessary to consider more projections, e.g. *all* the operations

$$K \rightsquigarrow \Delta_{\mu\nu}(K).$$

Here μ and ν are disjoint equicardinal *subsets* of $\mathcal{V}$, and the operation is again to replace, whenever feasible, μ by ν. (Thus now the defining bijection $\Delta^K_{\mu\nu} : K \to \Delta_{\mu\nu}(K)$ takes σ to σ unless $\nu \subset \sigma$, $\mu \cap \sigma = \emptyset$ and $(\sigma \setminus \nu) \cup \mu \notin K$ when it maps σ to $(\sigma \setminus \nu) \cup \mu$.)

A simplicial set K is said to be *compressed* with respect to a given total order of $vert(K)$, if it is invariant under all these operations $\Delta_{\mu\nu}$, with $\mu, \nu \subset vert(K)$ and $max(\mu) < max(\nu)$.

KRUSKAL-KATONA THEOREM. *Each non-negative integral vector $f = (f_0, f_1, \ldots, f_n)$ is the face vector of a unique compressed simplicial set K_f. Furthermore, f is the face vector of some simplicial complex only if K_f is a simplicial complex.*

Proof. The first part follows because, if the set of i-simplices of K is invariant under all $\Delta_{\mu\nu}$,with $\mu, \nu \subseteq vert(K)$ and $max(\mu) < max(\nu)$, then it must in fact consist of the first f_i, i-simplices from $vert(K)$, under the *total* order $\sigma <_{AL} \theta \iff max(\sigma \Delta \theta) \in \theta$.

To see the second part note[3] that the operations $\Delta_{\mu\nu}$ preserve face vectors, and have the following two properties:

(a) $$L \subset K \implies \Delta_{\mu\nu}(L) \subset \Delta_{\mu\nu}(K).$$

(b) $$\Delta_{\mu'\nu'}(K) = K \ \forall (\mu', \nu') \subset (\mu, \nu) \implies \partial\Delta_{\mu\nu}(K) \subseteq \Delta_{\mu\nu}(\partial K).$$

Using (a) and (b) it follows that, if the operations $\Delta_{\mu\nu}$,with $\mu, \nu \subseteq vert(K)$ and $max(\mu) < max(\nu)$, are performed in such an order that $|\mu| = |\nu|$ is non-decreasing, then the simplicial complex property $\partial K \subseteq K$ is preserved at each stage, resulting finally in a simplicial complex which is compressed and has the same face vector as K. **q.e.d.**

[3] Though (a) is trivial, the verification of (b) is much more messy than the analogous property of Th.1 of §3.

REMARKS. (1) *There is an analogous projection, also preserving face vector, onto simplicial sets K which, with respect to some total order of $vert(K)$, are invariant under all $\Delta_{\mu\nu}$ with $\mu, \nu \subset vert(K)$ and $min(\mu) < min(\nu)$.*

Such a K is initial with respect to the *lexicographic* total order $\sigma <_L \theta \iff min(\sigma \Delta \theta) \in \theta$ on equicardinal subsets of $vert(K)$. This lexicographic order, as well as the above anti-lexicographic order, are both bigger than the product partial order.

(2) *It is also useful to consider* partially compressed *simplicial sets.*

In other words there is some partial order $\leq$ on $vert(K)$ and we require that K be invariant under all operations $\Delta_{\mu\nu}$ with $\mu, \nu \subset vert(K)$ and, say, $min(\mu) \subset min(\nu)$.

Of course this condition is weak or strong depending upon the coarseness of the partial order; in case it is just $=$, any K is partially compressed !

A theorem analogous to the one above still holds, except that one can no longer assert the uniqueness of the compression.

Now, for each *idempotent* $e : \mathcal{V} \to \mathcal{V}$, $e^2 = e$, we introduce the operation

$$\mathcal{B} \rightsquigarrow \Delta_e(\mathcal{B})$$

on semi-simplicial sets, which replaces a semi-simplex ξ by $e \circ \xi$ whenever feasible. (Thus the defining bijection $\Delta_e^{\mathcal{B}} : \mathcal{B} \to \Delta_e(\mathcal{B})$ takes ξ to ξ, unless $e \circ \xi \notin \mathcal{B}$ when it maps ξ to $e \circ \xi$.)

And, similarly, also an operation

$$B \rightsquigarrow \Delta_e(B)$$

for sets of monomials or oriented semi-simplices $[\xi]$, which replaces $[\xi]$ by $[e \circ \xi]$ whenever feasible.

Note that the above 2 operations preserve the face vectors of the semi-simplicial sets $\mathcal{B}$ and B.

REMARKS. (3) Thinking of a simplex as the *support* of a semi-simplex, each idempotent $e : \mathcal{V} \to \mathcal{V}$ also specifies an operation

$$K \rightsquigarrow \Delta_e(K)$$

on simplicial sets which replaces a simplex $supp(\xi)$ by $supp(e \circ \xi$ whenever feasible.

This operation only preserves the total number of simplices of K.

(4) *This last operation on simplicial sets, which depends only on the disjoint sets* $\nu = \nu_e = \{v : e(v) \neq v\}$ *and* $\mu = \mu_e = e(\nu)$, *is, even for the case* $|\mu| = |\nu|$, *different from the operation* $\Delta_{\mu\nu}$.

To always preserve the face vector it is necessary to *restrict* the allowable replacements further, as we did while defining $\Delta_{\mu\nu}$.

These restrictions can also be described by saying that we think of a simplex σ as a *nonzero* decomposable exterior form, and replace it iff the form $e \circ \sigma$ is also nonzero, and not already in K. This viewpoint immediately suggests another, still more restrictive, operation in which one also demands that the linear independence of the forms be preserved. This will be pursued in §7.

A *compressed semi-simplicial complex* $\mathcal{B}$ is one which, with respect to some given total order of its vertices is invariant under all Δ_e with $\nu_e, \mu_e \subset vert(\mathcal{B})$ and $max(\mu_e \Delta \nu_e) \in \nu_e$. Likewise, for a monomial set B.

MACAULAY'S THEOREM. *Each non-negative integral vector* $h = (h_0, h_1, \dots)$ *is the face vector of a unique compressed set of monomials* B_h. *Furthermore,* h *is the face vector of an order ideal of monomials only if* B_h *is an order ideal of monomials. Likewise for semi-simplicial sets and complexes.*

Proof. ... ... **q.e.d.**

We now discuss the connection of the above result for semi-simplicial complexes with the theorems of Clements-Lindström, Bollobàs-Leader, Harper, etc.

Finally we look at the numerical versions of Kruskal-Katona and Macaulay's theorems.

§7. Linear shifting.

We will first look at some generalities, and then study one particular such shifting operation in more detail.

DEFINITION.

We already pointed out in §6, Remark (4), that it might be useful to consider shifting operations in which linear independence over some chosen *field of coefficients* F is chosen as a criterion for allowing replacements.

On all semi-simplicial sets such a projection

$$\mathcal{B} \rightsquigarrow \Delta_{\mathsf{F}}(\mathcal{B})$$

can be defined as follows.

Let $\mathcal{L}(\mathcal{B})$ be the F-vector space spanned by the semi-simplices of $\mathcal{B}$.

Next, let V be the subspace $\mathcal{L}(min(\mathcal{B}))$ spanned by the *minimal* nonempty simplices of $\mathcal{B}$. Note that $\mathcal{B}$ is a complex if and only if $min(\mathcal{B}) = vert(\mathcal{B})$.

Now take a totally ordered graded basis $x_1, x_2, \dots$ of V with dimensions non-decreasing.

Consider now all *words* in these x_i's. One can think of these as members of $\mathcal{L}(\mathcal{B})$ in the obvious way, and as such they clearly span this vector space.

With respect to the lexicographic ordering of the words, we pick, out of this spanning set, the first vector space basis of $\mathcal{L}(\mathcal{B})$, and that is called $\Delta_{\mathsf{F}}(\mathcal{B})$.

VARIATIONS.

(1) Alternatively the same semi-simplicial set can be obtained from the above set of generators by a *sifting* (or sieving) process: cross out any word which is a linear combination of the lexicographically preceding words.[4]

(2) Clearly the definition depends upon the basis $x_1, x_2, \dots$ chosen. But is true that, if we take care to choose a *generic basis,* then the result is independent of this choice upto a semi-simplicial isomorphism. Here by generic we mean that if the x_i's are expressed in terms of the *canonical basis* $\mathcal{B}$ of $\mathcal{L}(\mathcal{B})$, then the coefficients occuring in the transformation matrix are algebraically independent over the prime subfield F_p, $p = char(\mathsf{F})$, of F. A sufficient condition for the existence of such generic bases is that F be pretty *big* , i.e. it should have a large enough transcendence degree over its prime subfield.

(3) We have given a one-step definition of $\Delta_{\mathsf{F}}(\mathcal{B})$ (like that of K_f in the proof of the Kruskal-Katona theorem). This has obvious advantages. But clearly it is of interest to know if one can again define (independent of order considerations!) some *elementary shifting operations* utilizing linear independence, and then characterize $\Delta_{\mathsf{F}}(\mathcal{B})$ by its invariance with respect to those of these operations which obey some order-theoretic

[4]Thus *shifting* resembles e.g. the sieve of Eratosthenes. Possibly both fit neatly, together with *Bernouilli shifts*, etc., into a single more general framework?

conditions. We'll look at this problem in more detail for the particular operation which will be considered below.

(4) Sometimes it is useful to only *partially order* the x_i's, and then pick a spanning set initial with respect to its lexicographic extension to words. Of course now in general $\Delta_{\mathsf{F}}(\mathcal{B})$ is neither unique nor a basis, but in some cases, some particular feature (see e.g. §8 which deals with equivariant linear shifting) still guarantees both, and then such a variation is especially good.

(5) The way, considered 'obvious' above, for regarding a word in the x_i's as a member of $\mathcal{L}(\mathcal{B})$, was to expand it out in terms of the vertices of $\mathcal{B}$, and keep only the part which is a linear combination of semi-simplices of $\mathcal{B}$. This entails thinking of $\mathcal{L}(\mathcal{B})$ as the vector space of all *cochains* of $\mathcal{B}$. There is a *dual shifting operation* which involves thinking of this F-vector space as that of *chains* of $\mathcal{B}$. As such, it is now a sub(rather than a quotient)space of the space spanned by all semi-simplices in the vertices. Now, for each element of $\mathcal{L}(\mathcal{B})$, one considers the lexicographically *first word* in the support of the linear combination of words equalling this element. Then it can be verified that $\Delta_{\mathsf{F}}(\mathcal{B})$ is precisely the set of all first words of $\mathcal{L}(\mathcal{B})$.

(6) For *sets of monomials* we have thus an operation

$$B \rightsquigarrow \Delta_{\mathsf{F}}(B),$$

utilizing the subspace $L(B)$ of order-preserving monomials, and the commutative multiplication of polynomials, rather than the product defined by word juxtaposition.

(7) One can think of (ordinary) simplices as oriented semi-simplices with no vertex repetitions, i.e. as *exterior forms* . Again the definitions of this linear shifting, dealt with in more detail below, are similar.

(8) The notion of linear shifting makes sense for (a given 'canonical basis' of) any F-algebra, i.e. a vector space equipped with a *multilinear associative product*. So far we have referred to the products defined by word juxtaposition, polynomial multiplication, and exterior multiplication. For the use of yet another product see §8.

With the general methodology now set out, we work out one particular example in complete detail. However note that many of the definitions, assertions, and proofs given below apply also to other cases.

Exterior Shifting

This operation

$$K \rightsquigarrow \Delta_{\mathsf{F}}(K)$$

on simplicial sets is defined as follows.

The cochains of a simplicial complex K, with coefficients in F, constitute an exterior F-algebra $C^*(K)$, *viz.* that generated by $vert(K)$, *mod* the ideal generated by decomposables not supported on simplices of K.

[Regarding products, note that word juxtaposition defines the *tensor product* in $\mathcal{L}^*(K)$. This of course does not restrict to a product on the commutative cochains $L^*(K)$. So one symmetrizes it to get the *symmetric product* or polynomial multiplication. Unfortunately there are, on the vector spaces occuring in its decreasing filtration,

$$\cdots\cdots \supset L^*_{[2]}(K) \supset L^*_{[1]}(K) = C^*(K),$$

(here $L^*_{[r]}(K)$ consists of those commutative cochains c such that $c(\xi) = 0$ whenever a vertex repeats more than r times in the semi-simplex ξ) no suitable products for $r > 1$.]

When K is only a simplicial set, then $C^*(K)$, as just described, is merely an F-vector space.

Choose any totally ordered (with dimension non-decreasing) generic basis $x_1, x_2, \ldots, x_N$ of the F-vector space V spanned by $min(K)$, and note that each of its words $x_{i_1}x_{i_2}\ldots$ determines the element $[x_{i_1} \wedge x_{i_2} \wedge \cdots]$ of $C^*(K)$. As such the set of all words spans $C^*(K)$.

But, words with vertex-repetitions give zero, and permutations of the same letters give elements differing at most upto sign. So, any vector space basis of $C^*(K)$, contained in this spanning set, identifies naturally with a simplicial set having the same face vector as K.

With respect to the lexicographic total order on words, $\Delta_{\mathsf{F}}(K)$ is the least such basis.

[One can similarly shift any B by using $L^*(B)$. But, if the semi-simplices of B have vertices repeated at most r times, the lack of a suitable product stands in the way of defining a a more restrictive process using $L^*_{[r]}(B)$, which would result in a semi-simplicial set $\Delta_{\mathsf{F}}(B)$ having the same face vector, and the same bound r on its vertex-repetitions.]

We now plan to equip this operation with a suitable linear isomorphism

$$\Delta^K_{\mathsf{F}} : C^*(K) \to C^*(\Delta_{\mathsf{F}}(K)).$$

This will be a composition of some maps:

(D) $$D : C^*(K) \to C^*(K).$$

If K is only a simplicial set we might as well define D to be the identity map. However for simplicial complexes, it is better to define D to be the linear isomorphism which multiplies each nonempty simplex $\sigma \in K$ with the nonzero number $(x_1)^{|\sigma|}(\sigma)$. The reason being the following.

(a) *If K is a complex, then D is an algebra automorphism obeying*

$$D \circ \delta = (x_1 \wedge) \circ D.$$

Proof. To see this first note that the minimal elements $vert(K)$ generate this algebra, and the value of D on any σ equals the product of the values $x_1(v)$ on its vertices v.

Secondly note that the (ordinary) *coboundary δ is the same as taking wedge with the the sum of all the vertices* . Under D, this sum changes to x_1. *q.e.d.*

REMARKS. (1) The above suggests a *generalized cohomology* :

For the coboundary operator $\delta_{[q]}$ take wedge with the sum of all the cardinality q simplices of K. One has

$$\delta_{[q]} \circ \delta_{[q]} = 0$$

for all odd $q > 0$, and, if field characteristic is 2, for all $q > 0$.

For each q there is an analogous $D_{[q]}$ which converts $\delta_{[q]}$ into taking wedge with the first cardinality q simplex of $\Delta_{\mathbf{F}}(K)$.

(2)*However it seems that there must be some loss of genericity if one wants a 'D' which simultaneously transforms* all *these $\delta_{[q]}$ in this manner.*[5]

(3) Note that the linear automorphism $1 + \delta_{[1]} + \delta_{[2]} + \cdots$ is of *order* 2 .Thus (2) amounts to asking for an 'internal' $\mathbb{Z}_2$-shifting operation as against the 'external' $\mathbb{Z}_2$-shifting operation of §8.

(4) In the above proof we only used the *ellipticity* of the first vertex x_1, i.e. that it is a linear combination of the vertices of K with all coefficients nonzero. Taking wedge with any elliptic cardinality q simplex likewise yields the same cohomology as that given by $\delta_{[q]}$.

(L) $$L : C^* \to C^*(\Delta_{\mathbf{F}}(K)).$$

[5] We will do this later in §11, using the supersymmetry ideas of Witten.

Note that any linear combination of the words in the x_i's determines an element of $C^*(K)$, as well as that of $C^*(\Delta)$, $\Delta = \Delta_{\mathsf{F}}(K)$. While defining L we want to carefully distinguish between these two elements, so we'll use the suffixes K and Δ respectively.

Again, if K is just a simplicial set we might as well define L to be the obvious map $\sigma_K \mapsto \sigma_\Delta$ for all $\sigma \in \Delta$. However if it is a complex, it is better to add a *correction term* $(c_\sigma)_\Delta$ whenever $x_1 \wedge \sigma \notin \Delta$. This is the linear combination of the simplices of Δ preceding σ such that $(x_1 \wedge \sigma)_K = (x_1 \wedge c_\sigma)_K$. The reason being the following.

(b) *The L just defined is a linear isomorphism obeying*

$$L \circ (x_1 \wedge) = (x_1 \wedge) \circ L.$$

Proof. That it is one-one, and so onto, follows from the fact that its matrix, with respect to the bases of the 2 spaces determined by Δ is lower triangular with ones on the diagonal.

The commutativity follows because of the correction term which was added while defining L. *q.e.d.*

REMARKS. (5) An analogous correction term will give an analogous linear isomorphism $L_{[q]}$ commuting with the coboundaries given by taking wedge with the first cardinality q simplex of Δ.

It becomes a little more involved when one wants a single 'L' well behaved for all q in this manner.

(6) However note that *in $C^*(\Delta)$, the first simplices are certainly not elliptic,* so one should not expect such an L to induce isomorphisms of generalized cohomologies. But all is not lost.

(c) *For any simplicial complex K, the composition $L_{[q]} \circ D_{[q]}$ induces a surjection in qth generalized cohomologies.*

Proof. This follows because for the complex Δ one has a spectral sequence, converging to the qth generalized cohomology, whose first term is the cohomology determined by taking wedge with its first q-simplex. *q.e.d.*

REMARKS. (7) This result will be used later in §11 to show that this shifting operation, which does *not* preserve generalized cohomologies, does at least preserve the Cohen-Macaulay property.

However an alternative proof of this result, which avoids the use of generalized cohomologies, is given in this section itself.

(8) But, for $q = 1$ (the case of ordinary cohomology with coefficients in F), *genericity* (whose full strength we have not used till now) does give us an induced cohomology isomorphism.

To show this we define a third map

(U) $$U : C^*(\Delta) \to C^*(\Delta).$$

This takes a simplex $\sigma \in \Delta$ to itself unless the first vertex lies in it, when we map it to the sum of all the simplices of Δ obtained by replacing this vertex by a higher vertex.

(d) *For any simplicial complex K, the simplicial complex $\Delta_{\mathsf{F}}(K)$ is shifted with respect to the total order of its vertices $x_1, x_2, \ldots$. Further, U is a linear isomorphism obeying*

$$\delta \circ U = U \circ (x_1 \wedge).$$

Proof. The formula follows easily from the first part. That U is one-one, and so onto, follows from the fact that its matrix, with respect to the basis provided by the simplices of Δ, is upper triangular.

[Before doing the first part note that we should have done before the verification (easy, once one thinks of shifting as "sifting": see **(1)** above) that Δ is indeed a simplicial complex.]

The point where genericity is used is that every partial order preserving map can be realized by an isomorphism of F over its prime subfield F_p, and so acts on linear dependencies. Since this action is compatible with the lexicographic order (which was used in this "sifting") it follows that Δ is closed under the product partial order, i.e. is a shifted complex. *q.e.d.*

REMARKS. (9) *Even for the fact that Δ is shifted, we did not use the full strength of genericity.*

For example if F is the field of all rational functions in N variables over its prime subfield F_p, then we have all the field automorphisms we need. (This is considerably less than the transcendence degree N^2 required for full genericity.)

The above choice of F links $C^*(K)$ with the *de Rham complex* of K, a connection discussed further in §11.

(10) *And, just for the commutation rule of* **(d)**, *one needs even less.*

The required replacement-by-the-first-vertex, or the *near-cone* property, is guaranteed by even less field automorphisms.

(11) However, for $q > 1$, the replacement-by-the-first-cardinality q simplex-property is *not* weaker than shifting.

So we don't have, for $q > 1$, a $U_{[q]}$ which obeys an analogue of the above result.

The aforementioned map $\Delta_{\mathbf{F}}^{K}$ is now defined to be the composition $U \circ L \circ D$.

It is useful also to consider the dual linear shifting within $C_*(K)$, as e.g. in the proof of the following.

(e) *Linear shifting of a simplicial set K does not increase the face vector of $K \wedge K$.*

Here $K \wedge K$ denotes *the join of K with itself*, i.e. consists of the joins of all disjoint pairs of simplices of K. Note that, unlike $K \cdot K$, which was the join of two disjoint copies of K, $K \wedge K$ has no useful $\mathbb{Z}_2$-action.

Proof. Consider any two disjoint simplices of $\Delta_{\mathbf{F}}(K)$. They are the first words of two elements of $C_*(K)$. So their join is the first word of the element of

$$C_*(K \wedge K) = (C_*(K)) \wedge (C_*(K))$$

given by taking the wedge product of these two elements.

So we have in fact checked

$$\Delta_{\mathbf{F}}(K \wedge K) \supseteq (\Delta_{\mathbf{F}}(K)) \wedge (\Delta_{\mathbf{F}}(K)).$$

from which the result follows at once. *q.e.d.*

The proof given below of the next result is quite different from the original (see §11) which used generalized cohomologies.

KALAI'S THEOREM. *A simplicial complex K is Cohen-Macaulay or m-Leray over mod p coefficients $\mathbf{F}_p$, if and only if the associated generically shifted simplicial complex $\Delta_{\mathbf{F}}(K)$, $\mathbf{F} \supset \mathbf{F}_p$, is also Cohen-Macaulay or m-Leray.*

Note further that for $\Delta_{\mathbf{F}}(K)$ the Cohen-Macaulay and m-Leray properties have the simple combinatorial interpretations given before in Theorem 2 (c) and (e) of §3.

Proof. To establish 'only if' we proceed as follows:

Cochain isomorphisms analogous to the above 'diagonal' and 'lower-triangular' maps D and L can be defined in the bigger cochain complexes $L^*_{[r]}$.

$$L^*_{[r]}(K) \xrightarrow{D} L^*_{[r]}(K) \xrightarrow{L} L^*_{[r]}(\Delta_{\mathbf{F}}(K)).$$

Here D is again defined by multiplying vertices by the value of x_1 and L like before, except that every vertex of the correction term is assigned same multiplicity as the corresponding vertex of the basis word being considered.

[However note that these maps do not extend the previous: a word in the vertices having a certain maximum multiplicity, when written in terms of words in the x's might need more multiplicity.]

On the other hand there is no 'upper-triangular' U now, since the order ideal of monomials $B_{[r]}(K)$ is not shifted. But in any case there is an obvious spectral sequence from the $x_1\wedge$ cohomology of this to its δ cohomology.

But we saw in §5 that, for r even and large, this coincides with the *total cohomology* of K, i.e. the direct sum of the cohomologies at all the links.

So the total cohomology of $\Delta_{\mathbb{F}}(K)$ is no more than that of K, in particular it must vanish with that of the latter.

[Since the converse 'if' direction is not so important now, we'll add its proof later on.] **q.e.d.**

§8. Equivariant shifting.

Within the class of free G-simplicial complexes, $G \neq 1$, there does not seem[6] to be any nice projection, onto a combinatorially much simpler class, which preserves equivariant (co)homology. However we will show that, in a somewhat bigger class, there are indeed very pleasant operators of this kind.

For the sake of simplicity we will restrict ourselves to the case $G = \mathbb{Z}_2$. It seems however that the method is absolutely general, and will work for any group G.

The objects which we will be shifting are free $\mathbb{Z}_2$-semi-simplicial complexes $\mathcal{E}$ with no vertex repetitions, and with no semi-simplex containing both a vertex and its antipode, and with only at most as much sign-commutativity as is compatible with the group action.

[Note however that, unlike simplicial complexes, there are *free* $\mathbb{Z}_2$-semi-simplicial complexes $\mathcal{E}$, with some semi-simplices containing both a vertex and its antipode.]

First we show that going to this bigger class does not lose the information in which we are interested.

So let E be a free $\mathbb{Z}_2$-simplicial complex. We label the vertices of each antipodal pair by the group elements: one of them will be called of the first, and the other of the second, type.

If $\mathcal{E}$ is the semi-simplicial complex consisting of all semi-simplices, without vertex-repetitions, supported on the simplices of E, then the

[6]but see §8 bis.

above typing of its vertices can be extended to all its semi-simplices in the following natural manner.

A semi-simplex ξ of $\mathcal{E}$ is said to be of *type* $a = a_1, \overline{a_2}, a_3, \ldots$ if its first a_1 vertices are of the first kind, next a_2 of the second, and so on. Likewise we have types $\overline{a} = \overline{a_1}, a_2, \overline{a_3}, \ldots$. Clearly the obvious $\mathbb{Z}_2$ action on $\mathcal{E}$ is free, the antipode $\overline{\xi}$ being of the antipodal type $\overline{a}$.

[In fact had we started from an E (say $K \cdot K$) for which the $\mathbb{Z}_2$-action is free only on the vertices, even then the resulting $\mathcal{E}$'s $\mathbb{Z}_2$-action would be free. Now its simplices could have contained both a vertex and its antipode.]

For some purposes it is more convenient to go to an oriented quotient $(\mathcal{E})_o$ of $\mathcal{E}$. This is obtained by declaring two semi-simplices of $\mathcal{E}$ as having the same (*resp.* opposite) $\mathbb{Z}_2$*-orientation* if each can be obtained from the other by an even (*resp.* odd) number of transpositions of vertices of the same type.

THEOREM 1. PROPERTIES OF $E \rightsquigarrow \mathcal{E}$.

(a) *Any van Kampen obstruction class* $\mathfrak{o}$ *of* E *vanishes if and only if the corresponding class* $\mathfrak{o}$ *of* $\mathcal{E}$ *also vanishes.*

Proof. The quotient map $\mathcal{E} \to E$ commutes with the $\mathbb{Z}_2$-action.

Next, choose any total ordering of the pairs of antipodal vertices of E. Then there is a $\mathbb{Z}_2$-map $E \to \mathcal{E}$, mapping any simplex of E (which contains at most one member from each pair of antipodal vertices) to the total ordering of its vertices determined by this choice.

The result follows immediately from the existence of these continuous equivariant maps. *q.e.d.*[7]

(b) *The number of oriented semi-simplices of* $\mathcal{E}_0$*, of type* $a_1, \overline{a_2}, a_3, \ldots$ *or* $\overline{a_1}, a_2, \overline{a_3}, \ldots$*, coincides with the number of simplices of* E *which have* $a_1 + a_3 + \cdots$ *vertices of one type, and* $a_2 + a_4 + \cdots$ *of the other.*

Proof. This follows because each one of our $\mathbb{Z}_2$-oriented simplices ξ is uniquely determined (upto sign) by its type a, and its two *parts*: i.e. the (ordinary) simplices ξ_1 and $\overline{(\xi_2)}$ made up of all its letters of the first and second type. *q.e.d.*

Note that one has $\xi_1 \cap \xi_2 = \emptyset$, and that the antipodal simplex $\overline{\xi}$ has type $\overline{a}$ and parts $(\overline{\xi})_1 = \xi_2$ and $(\overline{\xi})_2 = \overline{(\xi_1)}$.

We will be referring to the number of terms of a as the *length* of the type a.

[7] Note however that the $\mathbb{Z}_2$-cohomology of $\mathcal{E}$ is generally much bigger than that of E. Had we allowed vertex-repetitions this too would have been preserved.

The next property pertains in fact to the join functor $K \rightsquigarrow K \cdot K$.

(c) *The number of simplices (σ, θ) of $K \cdot K$ with $|\sigma| = i$ and $|\theta| = j$ and $\sigma \cap \theta \neq \emptyset$ is at most $j \cdot f_{i-1}(K) \cdot f_j(K)$.*

Proof. Deleting one of the $r \geq 1$ vertices of the first factor which are also in the second, we get r simplices of $K \cdot K$, with cardinality of the first factor, as well as that of its intersection with the second, one less.

Conversely at most $j - r + 1$ of the original kind of simplices of $K \cdot K$ can be recovered from this new kind, by adding one of the vertices of the second factor, not already in the first, to the first factor.

The result follows because $\frac{j-r+1}{r} \leq j$. *q.e.d.*

In the following we will use the notation $\mathcal{E}$ for the oriented quotient $\mathcal{E}_0$ instead.

We will now discuss an equivariant shifting operation

$$\mathcal{E} \rightsquigarrow \Delta_{\mathsf{F}}(\mathcal{E}),$$

determined by linear independence in the $\mathbb{Z}_2$-space $L^*(\mathcal{E})$ of oriented cochains of $\mathcal{E}$ with coefficients from the field F. The antipode of a cochain c will be denoted by $\overline{c}$.

For V, the F-vector space spanned by $vert(\mathcal{E})$, we choose a $\mathbb{Z}_2$-basis $x_1, \overline{x_1}; x_2, \overline{x_2}; \ldots; x_N, \overline{x_N}$. This is given the *partial order* $\leq$ with just two maximal chains, each of length N, *viz.* the displayed sequences of letters of the first and second type. Also we take care that the x_i's are linear combinations only of vertices of $\mathcal{E}$ of the first type, and so $\overline{x_i}$'s are the corresponding identical linear combinations of vertices of the second type. Except for this equivariance condition, we will assume that the coefficients occuring in the transformation matrix are algebraically independent over the prime subfield F_p. As before, this *equivariant genericity* is possible to arrange, provided the field F is big enough.

Consider now all words, i.e. finite sequences, from this basis. Each determines an element of $L(\mathcal{E})$. These span this vector space. Note that words containing both a letter and its antipode, need not determine a zero element. But we delete these anyway. [It can be checked that the remaining words still span this vector space.] → ? Note for these that the element determined by a word changes sign if we make a transposition of two letters of the same type, so in particular if a letter repeats, this element is zero.

We now turn to the definition of $\Delta_{\mathsf{F}}(\mathcal{E})$. To do this we take the lexicographic extension $<_L$ of our partial order, of letters, to all words. Note that this is a partial order which compares two sequences iff their first distinct entries are letters of the same type. We now delete all words which determine elements depending linearly on elements determined by $<_L$-smaller words. What remains is $\Delta_{\mathbf{F}}(\mathcal{E})$.

Á priori, this is only a spanning set. In fact it is a basis:

THEOREM 2. PROPERTIES OF $\mathcal{E} \rightsquigarrow \Delta_{\mathbf{F}}(\mathcal{E})$.

(a) $\Delta_{\mathbf{F}}(\mathcal{E})$ *is a free* $\mathbb{Z}_2$*-semi-simplicial complex having the same number of semi-simplices of each type as* $\mathcal{E}$.

Proof. To see this extend the typing of letters to words, i.e. a word will be of type a (likewise $\overline{a}$) if its first a_1 letters are of first type, next a_2 of the second type, etc. Note that the first distinct letters of words of the same type are of the same type: so $<_L$ *restricts to a total order on each type.*[8]

Next, note that types give a direct sum decomposition of $L^*(\mathcal{E})$, and any element of our spanning set lies in one of these summands.

So $\Delta_{\mathbf{F}}(\mathcal{E})$ is indeed a basis of $L^*(\mathcal{E})$, and has the same number of semi-simplices of each type as $\mathcal{E}$.

That it is a complex follows again from the fact that the basis was obtained by a sifting process. (?)] crucial

Since sifting of antipodal types is identical it follows that it has a free $\mathbb{Z}_2$-action. *q.e.d.*

Note that the partial order of letters induces the *product partial orders* on words of the same type having disjoint first and second parts.

(b) $\Delta = \Delta_{\mathbf{F}}(\mathcal{E})$ *is* $\mathbb{Z}_2$-shifted *: i.e. closed with respect to the product partial order on the disjoint words in each type.*

Proof. Assume that a disjoint word ξ of a certain type is in Δ, and another η, disjoint and of the same type, and less than ξ in the product partial order, is given.

If possible, suppose that the element determined by η is linearly dependent on elements determined by $<_L$-smaller words of the same type.

Choose a field automorphism of $\mathbf{F}$ over $\mathbf{F}_p$, which induces a $\mathbb{Z}_2$-vector space isomorphism, moving the letters occuring in ξ, in order, to those occuring in η, and the remaining ones, in order, to the remaining ones.

Then the $\mathbf{F}_p$-linear map of $L^*(\mathcal{E})$, determined by this automorphism, when applied to the above linear dependency, will exhibit the element determined by ξ as a linear combination of elements determined by $<_L$-smaller words of the same type, which is not possible. *q.e.d.*

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[8] This would not be true for the smaller partial order which demands that the two parts ξ_1 and ξ_2 be both less than η_1 and η_2.

(c) *There is a linear type-preserving $\mathbb{Z}_2$-cochain isomorphism*

$$\Delta^{\mathcal{E}}_{\mathsf{F}} : L^*(\mathcal{E}) \to L^*(\Delta_{\mathsf{F}}(\mathcal{E})).$$

(?)

Proof. The construction which parallels that considered in §7 proceeds as follows:

$$D : L^*(\mathcal{E}) \to L^*(\mathcal{E}).$$

This 'diagonal' map is an algebra isomorphism which multiplies each vertex of $\mathcal{E}$, depending upon its type, with the nonzero value of either x_1 or $\overline{x_1}$ at that vertex.

The coboundary δ is the sum of a number of operators well-behaved with respect to the type decomposition. More precisely if type b is same as type a except in one (possibly empty) slot, where it is one more, then there is a summand of δ running from types a to b, working in that slot just like the ordinary coboundary.

It is clear that D will change this to taking wedge with either x_1 or $\overline{x_1}$ in this slot.

Denoting the sum of all these operators by $(x_1 + \overline{x_1})\wedge$ we can write briefly

$$D \circ \delta = ((x_1 + \overline{x_1})\wedge) \circ D.$$

The next step is to define the 'lower-triangular'

$$L : L^*(\mathcal{E}) \to L^*(\Delta).$$

To do this note that there is a natural multilinear surjection from types of length one to other types given by

$$\{\sigma_1, \sigma_2, \sigma_3, \ldots\} \mapsto \sigma_1 \cdot \overline{\sigma_2} \cdot \sigma_3 \cdots .$$

These multiplications commute with taking wedge product in any slot. In a length 1 type we define L as before, and then, using the above multiplication, set

$$L(\sigma_1 \cdot \overline{\sigma_2} \cdot \sigma_3 \cdots) = L(\sigma_1) \cdot L(\overline{\sigma_2}) \cdot L(\sigma_3) \cdots$$

It follows that

$$L \circ ((x_1 + \overline{x_1})\wedge)_{\mathcal{E}} = ((x_1 + \overline{x_1})\wedge)_{\Delta} \circ L.$$

The wedge on the right side does not have an 'elliptic' character with respect to the vertices of Δ. But now we use the $\mathbb{Z}_2$-shifted property[9] of Δ to define, much the same way as L was, an 'upper-triangular' map

$$U : L^*(\Delta) \to L^*(\Delta)$$

which obeys the commutation rule

$$U \circ ((x_1 + \overline{x_1})\wedge)_\Delta = \delta \circ U.$$

The required cochain isomorphism is the composition $U \circ L \circ D$. *(??)* *q.e.d. (??)*

(d) *Thus, there is an induced isomorphism from the $\mathbb{Z}_2$-cohomology of $\mathcal{E}$ to that of $\Delta_{\mathsf{F}}(\mathcal{E})$, which* (for the case $char(\mathsf{F}) = 2$) *maps each van Kampen obstruction class $\mathfrak{o}$ of $\mathcal{E}$ to the corresponding characteristic class of $\Delta_{\mathsf{F}}(\mathcal{E})$.*

Proof. This follows from the Richardson-Smith exact sequence definition of these characteristic classes (which also showed that these classes are zero unless the field characteristic is 2). **q.e.d.**

Had we not thrown out, while sifting, the words having a vertex and its antipode both, then the above result is not true and the shifted semi-simplicial complex might very well have a nonzero $\mathfrak{o}$ even when that of the original was zero.

Now it remains only to study the combinatorially much simpler semi-simplicial complexes Δ.

THEOREM 3. $\mathbb{Z}_2$-SHIFTED SEMI-SIMPLICIAL COMPLEXES. *Let $\mathcal{E}$ be $\mathbb{Z}_2$-shifted.*

(a) *The Flores' sphere is $\mathbb{Z}_2$-shifted.*

From this it follows that the join of some Flores' spheres is also $\mathbb{Z}_2$-shifted. Likewise one sees even more easily that the octahedral spheres are also $\mathbb{Z}_2$-shifted.

Proof. In each type, with the number of both types of vertices $\leq (n+1)$, consider the $\mathcal{E}_0$ consisting of all semi-simplices, with the 2 parts disjoint, which determine a subsequence of $1, 2, \ldots, 2n+3$. This is a simplicial $\mathbb{Z}_2$-sphere simplicially isomorphic to the Flores' sphere $((\sigma)_n^{2n+2})_*$.

To see this note that the injection of the last named deleted join, determined by the order, is a bijection, because the number of such typed subsequences coincides precisely with the number of simplices in

[9] In fact a weaker $\mathbb{Z}_2$-near-cone property suffices for this purpose.

the deleted join. For example in top dimension the number of such typed subsequences is $2n+3$ in each of the $\binom{2n+2}{n+1}$ types, making required total $\binom{2n+3}{n+2} \cdot (n+2)$, the number of top simplices in the deleted join. *q.e.d.*

The question arises whether there are $\mathbb{Z}_2$-shifted spheres other than octahedral spheres and joins of $(\sigma^{2s}_{s-1})_*$'s ? For example, is the join-irreducible simplicial 4-sphere $(\mathbb{R}P^2_6)_*$ also $\mathbb{Z}_2$-shifted ? [Here $\mathbb{R}P^2_6$ denotes the minimal triangulation of the real projective plane, i.e. *one-half* of the icosahedron.] We'll see below that the answer to these questions is negative (?) and there are no shifted $\mathbb{Z}_2$-spheres other than the ones above.

If any of the above examples sits inside $\mathcal{E}$, in such a way that it is closed with respect to the product partial orders, then its highest dimensional simplices will occur with types $\geq 2r$, where r denotes the number of join-irreducible components.

The above observation generalizes as follows.

(b) *A $\mathbb{Z}_2$ m-dimensional sphere of $\mathcal{E}$ contains m-simplices of all types of length equal to or bigger than $2r$, for some r, and is then a join of r join-irreducible $\mathbb{Z}_2$-shifted spheres.*(?)

Proof. ...

....

The following now settles **(a)** .

(c) *If a van Kampen obstruction class* $\mathfrak{o}$ *is nonzero, then $\mathcal{E}$ must contain a $\mathbb{Z}_2$-shifted sphere of that dimension.* (?)

Proof. We sketch the argument only.

Pick a mod 2 equivariant cycle on which $\mathfrak{o}$ is nonzero.

We then check that it is a mod 2 sum of equivariant pseudomanifolds, so we can assume that our original mod 2 cycle was itself a pseudomanifold.

Using the $\mathbb{Z}_2$-shifting property we can then further assume that this pseudomanifold is closed with respect to the product partial orders.

The last and most difficult step is to show that such pseudomanifolds are automatically spheres. *q.e.d.* (?)

We will be establishing a (higher codimensional) *Heawood Inequality* with a *sharp* constant. However this requires a further analysis. So at this point it seems apt to give, especially in view of the fact that even the existence of *any* such inequality was moot for a very long time, an easier argument which yields a somewhat weaker result.

THEOREM 4. *Let K be a simplicial complex with $f_n(K) \geq (2n+3) \cdot f_{n-1}(K)$. Then K can not embed (even topologically) in $\mathbf{R}^{2n}$.*

Proof. (?) Put $E = K_*$, and then equivariantly shift the semi-simplicial complex $\mathcal{E}$ supported on E to a $\mathbf{Z}_2$-shifted semi-simplicial complex $\Delta = \Delta_{\mathsf{F}}(\mathcal{E})$, using a field F of characteristic 2.

It will suffice to show, under the given numerical condition, that the $(2n+1)$th van Kampen obstruction class $\mathfrak{o}$ of Δ is nonzero.

To see this note that the number of semi-simplices of Δ of type $n+1$ (or $\overline{n+1}$) equals those of $\mathcal{E}$ and so (by Th 2 (b)) equals the number $f_n(K)$ of n-simplices of K.

Consider now semi-simplices of Δ of type $n+1, \overline{n+1}$. The ones with their 2 parts intersecting (cf. Th 2 (c)) are at most $(n+1) \cdot f_{n-1}(K) \cdot f_n(K)$. And, the ones which are disjoint but contain a first vertex are less than $f_{n-1}(K) \cdot f_n(K)$ in number. The total count of both is thus less than $(n+2) \cdot f_{n-1}(K) \cdot f_n(K)$.

On the other hand the total number of semi-simplices of Δ of type $n+1, \overline{n+1}$ equals similar number for $\mathcal{E}$ which equals (Th 2 (b)) the number of $(2n+1)$-simplices of K_*, which (by Th 2 (c)) is at least $((2n+3)-(n+1)) \cdot f_{n-1}(K) \cdot f_n(K)$, i.e. bigger than the above count.

So Δ has a semi-simplex in type $n+1, \overline{n+1}$ whose 2 parts are disjoint, and which does not contain a first vertex.

Likewise there are analogous semi-simplices, with both parts disjoint, and not containing a first vertex, in any type a with $a_1 + a_3 + \cdots = \overline{a_2} + \overline{a_4} + \cdots = n+1$.

Consider now all disjoint semi-simplices which are less or equal to, in the product partial orders, than these semi-simplices. Because Δ is $\mathbf{Z}_2$-shifted all these are also in Δ.

But they form a $\mathbf{Z}_2$-subcomplex of Δ isomorphic to the Flores' sphere $(\sigma_n^{2n+2})_*$ whose $\mathfrak{o}$ is nonzero. **q.e.d.** (?)

The argument given above also shows that $\mathfrak{o}$ (of the associated free $\mathbf{Z}_2$-semi-simplicial-complex $\mathcal{E}$) is nonzero for any $\mathbf{Z}_2$-simplicial complex E, with $\mathbf{Z}_2$-action free on the vertices, provided the latter is sufficiently *top-heavy* in an obvious sense.

The argument for the sharp Heawood inequality will exploit the *product nature* of $E = K_*$.

THEOREM 5. *Let $\mathcal{E}$ be supported on the deleted join $E = K_*$ of an n-dimensional simplicial complex K, and let $\Delta(\mathcal{E})$ be its equivariant linear shift obtained by using a big field of characteristic 2.*

Then, if $\Delta(\mathcal{E})$ contains

$$(\sigma^{2s_1}_{s_1-1})\cdot\overline{\sigma^{2s_2}_{s_2-1}}\cdots, s_1+s_2+\cdots=n+1$$

in type $s_1, \overline{s_2}, \ldots$, then it must also contain its deleted join as a Flores' sphere.

Furthermore, the above can happen only if the corresponding van Kampen class is nonzero. (?)

REMARKS. (1) Here the words "in type" are essential. For example types of length 1, which constitute $\Delta(K)$, can contain $\sigma_0^2\cdot\sigma_0^2$ even if the graph K is planar.

(2) But the theorem does imply that *if an n-dimensional K embeds in $\mathbf{R}^{2n}$, then $\Delta(K)$ can not contain σ_n^{2n+2}*. This (or rather a straightforward generalization which is stated as the next theorem) was conjectured by Kalai.

(3) The last remark implies a *sharp Heawood inequality*:

If K^n embeds in $\mathbf{R}^{2n}$, then $f_n(K) < (n+2)\cdot f_{n-1}(K)$.

This follows because then $\Delta(K)$ contains σ_n^{2n+2}.

That the inequality is sharp follows by looking at the n-skeletons of cyclic $2n$-polytopes.

(4) Likewise the next result (*cf.* proof of Theorem 2 (f) in §3) immediately yields sharp Heawood inequalities for K^n's embedding in $\mathbf{R}^m$ for $m < 2n$.

(5) A converse of this theorem, which will be formulated as Theorem 7 below gives a very pleasant higher dimensional Kuratowski theorem.

Proof of Theorem 5. (?) For the sake of simplicity we give the argument only for case "in type $n+1$", the general case is more complicated only notationally.

In other words the hypothesis given is that σ_n^{2n+2} is contained in $\Delta(K)$ (cf. Remark (2) above). The required conclusion regarding its deleted join will follow from the following more general fact that we will establish.

For any 2 disjoint simplices σ and θ of $\Delta(K)$, the semi-simplex specified by the total ordering of $\sigma\cup\theta$, in each type having this length, is contained in $\Delta(\mathcal{E})$.

[Likewise the general case would generalize to an analogous type-restricted statement. Note that by an *Example* worked out before this shows for an irreducible T^n in $\Delta(K)$ that its deleted join is in. But not for, say, the Kuratowski 3,3 graph.]

To prove this we will use the dual "first word" definition of linear shifting which uses chains rather than cochains.

¿From the *free typed algebra* (obvious definition) to the free exterior algebra we have an obvious quotient map,

$$T(V \oplus \overline{V}) \rightarrow \Lambda(V).$$

The chains $L_*(\mathcal{E})$ form a subspace of the left side (generically situated with respect to the new x-basis of our free typed algebra), while the chains $C_*(K \wedge K)$ of the deleted join of K with itself form a similar subspace of the right side. (See (e) of §7.)

The kernel of the above quotient, and the inverse images of the 1-dimensional spaces spanned by the x-basis elements of $\Lambda(V)$, give a decomposition of the free typed algebra, and the x-basis of the latter, besides being of course compatible with its type decomposition, is also well-behaved with respect to this decomposition.

The image of $L_*(\mathcal{E})$ under the above map is $C_*(K \wedge K)$. Look at an inverse image in $L_*(\mathcal{E})$, in any type[10], of the element $\sigma \wedge \theta$ of $C_*(K \wedge K)$.

If we write it as a linear combination of the x-basis of the free typed algebra, obviously its first word corresponds to the total ordering of $\sigma \cup \theta$ in that type. Which establishes the desired assertion. **q.e.d.**

comment

Only a sketch follows.

Having selected the lexicographically first generic basis $\Delta(\mathcal{E})$ of $L(\mathcal{E})$, we now mod out by words whose two parts are not disjoint. Because of the product nature of our E, the remaining disjoint words, constitute a basis of this quotient algebra $\mathcal{A}$.

Denote these basis elements by $[\sigma_1 \cdot \overline{\sigma_2} \cdots]$. Using these basis elements we will now define a linear map

$$T : \mathcal{A} \rightarrow \mathcal{A}(\Delta(\mathcal{E})),$$

where the right side is also obtained by an analogous moding out of words with non-disjoint parts.

But note an important difference for the right side: now the disjoint words do not determine a basis of this quotient. For a word to survive till the quotient all its partitions must be supported on $\Delta(\mathcal{E})$. This point exclaims Remark (1) above.

T is defined first on words of length one as $U \circ L$, and then in general by

$$T[\sigma_1 \cdot \overline{\sigma_2} \cdots] = [T(\sigma_1) \cdot \overline{T(\sigma_2)} \cdots].$$

[10] ?exists?

The definition makes sense because of the above basis property. [But note that we have not asserted, and neither is it true, that it is a linear isomorphism.] Further, since we have already checked it for length one types, there is the following commmutation property:

$$T \circ ((x_1 + \overline{x_1})\wedge) = \delta \circ T.$$

Using the above remark regarding the necessity of all partitions to be supported, it follows that if a Kuratowski complex, and so its deleted join, occurs in the correct type, then it will be linearly independent in $\mathcal{A}(\Delta(\mathcal{E}))$.

The assertion regarding $\mathfrak{o}$ now follows by using T. *q.e.d.*

endcomment

THEOREM 6. *If K^n embeds in $\mathbb{R}^m$, $n \leq m \leq 2n$, then $\Delta(K)$ can not contain $\sigma_t^t \cdot \sigma_{s-1}^{2s}$ with $t + 2s = m + 1$.*

Proof. This is a straightforward generalization of the above, so the details are omitted. **q.e.d.**

The last result is the generalized Kuratowski theorem.

THEOREM 7. *A K^n has $(2n+1)$th van Kampen obstruction class $\mathfrak{o}$ zero if and only if $\Delta(\mathcal{E})$ does not contain*

$$(\sigma_{s_1-1}^{2s_1}) \cdot \overline{\sigma_{s_2-1}^{2s_2}} \cdots, s_1 + s_2 + \cdots = n + 1 \textit{ in type } s_1, \overline{s_2}, \ldots.$$

Proof. (?) Again we omit the details, but the key point is to use Th 3 (c) to obtain the 'only if' part. **q.e.d.** (?)

As the (?)'s point out there are many problems with the above execution of the basic idea, i.e. *equivariant shifting succeeds if done type-wise.* This idea itself however seems sound, so we will do §8 over again till the details get straightened out, leaving previous work standing so we can come back in case a wrong turn is taken.

§8 bis. Equivariant shifting.

This time we will stick to *simplicial* complexes only, and not use semi-simplices. However the idea of a 'typing', induced from some total order of the vertex-orbits, remains all-important. It seems that the following, once smoothened out, will generalize easily to all (or at least all finite) groups G. But, for now, we will continue to consider free $\mathbb{Z}_2$-complexes only, and we start out by looking at their combinatorics more carefully than before.

(A) FREE $\mathbb{Z}_2$-COMPLEXES.

Let $U = U_Y$ denote the *octahedral sphere*, i.e. the maximal free $\mathbb{Z}_2$-complex, on the union Y of the N *vertex-pairs* $\{v, \overline{v}\}$.

Vertices v will be called *positive* and the vertices $\overline{v}$ will be called *negative*. To conform to the notation being used, we refer to the involution as *conjugation*, so it 'changes the sign' of a vertex from positive to negative or conversely.

Note that each subset σ, of the set all our $2N$ vertices, is the disjoint union of its *positive part* σ_+ and *negative part* σ_-, obtained by separating its vertices according to their sign. The octahedral sphere U consists of all σ satisfying $\sigma_+ \cap \overline{\sigma_-} = \emptyset$.

For any simplicial complex $K \subseteq U$ the octahedral sphere determined by the vertices of K and their conjugates will be denoted by $U(K)$. At the other extreme, the minimal (free) $\mathbb{Z}_2$-complex of U containing K, i.e. $K \cup \overline{K}$, will be called $u(K)$.

For any simplicial complex $K \subseteq U$ we also define a (free) $\mathbb{Z}_2$-subcomplex $K_* \subseteq U$ as follows:

$$K_* = \{\sigma \cup \overline{\theta} : \sigma \in K, \theta \in K, \sigma \cap \theta = \emptyset\}.$$

Though the above formula is same as that used for the definition of *deleted join* before, note that we now allow the possibility that the conjugate simplicial complex $\overline{K}$ may intersect K. Thus the above definition is more general.

Note that this unary operation $K \rightsquigarrow K_*$ is nested between the commuting idempotents $u(K)$ and $U(K)$, and commutes with both.

The next construction depends on the choice of a *total order* on the set of all positive vertices. This induces a total order on the set of all vertex-pairs, and another on the conjugate set of all negative vertices. But note that the set of all the $2N$ vertices only gets a *partial order*.

However, since the vertex-pairs intersecting the positive and negative parts of any $\sigma \in U$ are disjoint, this partial order restricts to a total order on σ. Viewed under this total order, each nonempty simplex σ of U partitions into a finite sequence of nonempty subsets called its *slots*, such that

(i) all vertices in a slot have the same sign, and

(ii) the vertices of adjacent slots have opposite signs.

The *type* of $\sigma \in U$ is the corresponding integer sequence

$$type(\sigma) = \alpha = \cdots \alpha_i \, \overline{\alpha_{i+1}} \, \alpha_{i+2} \cdots$$

(here we write negative integers $-n$ as $\overline{n}$) which assigns to each slot the nonzero integer whose sign and absolute value coincide with the sign and cardinality of the slot.

Furthermore, in case this integral sequence happens to have an odd number of terms, we will make the convention that there is an unwritten symbol 0 or $\overline{0}$ in the end: thus the *length* of the type will be an even number ≥ 2. Accordingly, if the number of slots of σ is odd, we augment its slot factorization by an *empty (positive or negative) last slot.*

Conversely, for each finite sequence α of nonnegative integers having alternating signs, one has a subset U_α of U consisting of all simplices whose type is α. Thus the octahedral sphere U gets partitioned into these disjoint simplicial sets U_α.

For any simplicial complex $K \subseteq U$, we will denote by $U_\circ(K)$, the subcomplex of $U(K)$ which occurs as the following 'type-skeleton'.

$$U_\circ(K) = \cup_\alpha\{U(K) \cap U_\alpha : \alpha = type(\sigma), \sigma \in K\}.$$

We now associate, to any simplicial complex $K \subseteq U$, a (free) $\mathbb{Z}_2$-simplicial complex K^* as follows:

$$K^* = \{\varphi : \varphi \neq \sigma \cup \overline{\theta}, \sigma \in U_\circ(K) \setminus K, \theta \in U_\circ(K) \setminus K, \sigma \cap \theta = \emptyset\}.$$

When K and its conjugate $\overline{K}$ are disjoint we will refer to $K_* \cap K^*$ as the *reduced deleted join* of K. Note that the partial order on the set of our $2N$ vertices restricts, in general, only to a partial order on $vert(K)$. But, under the above disjointness hypothesis, it does restrict to a total order. Furthermore, there is a sign $+$ or $-$ attached to each vertex of K depending on whether it is positive or negative. The reduced deleted join of K is determined uniquely for any choices {total order , signs} on $vert(K)$.

The algebraic motivation for introducing this second unary operation $K \rightsquigarrow K^*$ will become clear in (D).

We end with some notation for types:

The *conjugate type* $\overline{\alpha}$ will be the integer sequence obtained by reversing all signs in α. Thus conjugate simplices have conjugate types.

A *partial order* $\preccurlyeq$ *on types* can be defined as follows.

Consider integer sequences in which 0 and $\overline{0}$ are interposed as equivalent to types, e.g. $\cdots 2\ \overline{0}\ 3\ \overline{1}\ 0\ \overline{5} \cdots$ will be equivalent to the type $\cdots 5\ \overline{6} \cdots$. We will say $\alpha \preccurlyeq \beta$ iff $\alpha_i \leq \beta_i$ for all i, possibly after some re-indexing and such interpositions.

Note that conjugate types are equivalent under this partial order and $\sigma \subseteq \theta$ implies $type(\sigma) \preccurlyeq type(\theta)$.

Lastly, we define an associative *binary operation* $*$ *on types* by juxtaposition followed by consolidation in case the abutting numbers have the same sign, so e.g. $(\cdots 5\,\overline{2}) * (\overline{3}\,4) * (\overline{1}\,6\cdots) = \cdots 5\,\overline{5}\,4\,\overline{1}\,6\cdots$.

(B) WEDGES.

In (A) we defined two unary operations 'lower and upper star' on subcomplexes of a universal free $\mathbb{Z}_2$-complex U, which associated to each subcomplex two free $\mathbb{Z}_2$-subcomplexes.

Analogous definitions can be made for *any* group G. However, to study the case $G = \mathbb{Z}_2$, the only other case needed (at least as motivation) is that of the trivial group $G = 1$, so we will introduce some distinguishing notation only for this case as follows:

Now our universal complex is the *closed simplex* S on the N vertices[11] v, and, for each subcomplex $K \subseteq S$, the closed simplex determined by the vertices of K will be called $S(K)$.

To any subcomplex K of S we associate another, called its *wedge* by

$$K_\wedge = \{\sigma \cup \theta : \sigma \in K, \theta \in K, \sigma \cap \theta = \emptyset\}.$$

Now $S_-(K)$ is the skeleton of the closed simplex S(K) determined by the *dimensions* occuring in K, i.e. if $dim K = n$, then $S_-(K)$ is simply the n-skeleton of $S(K)$. So, in this case, even the second operation $K^\wedge$, and thus the *reduced wedge* $K_\wedge \cap K^\wedge$, can be defined without using any total order.

$$K^\wedge = \{\varphi : \varphi \neq \sigma \cup \theta, \sigma \in S_-(K) \setminus K, \theta \in S_-(K) \setminus K, \sigma \cap \theta = \emptyset\}.$$

The algebraic motivation for introducing this second unary operation $K \rightsquigarrow K^\wedge$ will become clear in the following.

(C) EXTERIOR ALGEBRAS.

We choose, once for all, a field F *of coefficients.*

But we note that, later on, in order to ensure that generic shifting is possible, we will also assume that F is *big* over its characteristic subfield F_p, i.e. that it has a transcendence degree at least N over the latter.

And, for applications to embeddability questions, we will later on also assume that the field characteristic p equals 2 ; or, alternatively, that F is a characteristic zero field, big over the field of 2-adic numbers.

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[11] The cases $G = 1$ and $G = \mathbb{Z}_2$ will be consolidated by thinking of these as the positive vertices of U.

For our N-vertex closed simplex S, the vector space $\mathcal{C}(S)$ of all its *oriented chains* (i.e. linear combinations of oriented simplices), and the vector space $C(S)$ of all its *oriented cochains* (i.e. functions on oriented simplices whose values change sign with a reversal of orientation), have both as basis the set of all simplices of S, each equipped with some orientation. Choosing such a common basis gives an identification $\mathcal{C}(S) = C(S)$.

We now identify this vector space $\mathcal{C}(S) = C(S)$ further with the underlying vector space of the *free exterior algebra* Λ generated over F by the N vertices. As usual the product of this algebra will be denoted by $\wedge$, so we are speaking of the associative F-algebra with unity generated by the N vertices, subject to the relations

$$v \wedge w = -w \wedge v$$

for all vertices v and w.

As usual we *grade* this algebra by proclaiming the nonzero elements contained in the vector space spanned by the oriented (nonempty) simplices of a certain *dimension* , to be of 'degree' one more. And, as is usual for any graded algebra, any element will be called *decomposable* if it is a product of lower degree elements.

In particular, we note that each positive-dimensional oriented simplex of S, which has been identified with the corresponding wedge product of its vertices, is a decomposable whose sign changes with a reversal of orientation.

If V and W are vector subspaces of Λ, then $V \wedge W$ will denote the vector subspace spanned by all $\mu \wedge \nu$, $\mu \in V$, $\nu \in W$.

Using this we associate, to each graded vector subspace V of Λ, the graded vector subspace $V_\wedge = V \wedge V$. Note that if $V = V_0 + \cdots V_r + \cdots$, then the degree r summand $(V_\wedge)_r$ of $V_\wedge$ equals the (not necessarily direct) sum of the subspaces $V_s \wedge V_t$, as $\{s,t\}$ runs over all unordered pairs of non-negative integers such that $s + t = r$.

If V is a graded subalgebra or ideal, then so is $V_\wedge$.

So to each quotient graded algebra $A = \Lambda/I \cong \oplus_r \Lambda_r/I_r$, where $I = \sum_r I_r$ is a graded ideal, we can associate the quotient graded algebra $A^\wedge = \Lambda/I_\wedge$.

If K is a subcomplex of S, then the chains $\mathcal{C}(K)$ of K constitute a graded subspace of Λ, *viz.* that which is spanned by the oriented simplices of K. From this it follows at once that

$$(\mathcal{C}(K))_\wedge = \mathcal{C}(K_\wedge).$$

On the other hand, the cochains $C(K)$ of K form a quotient graded algebra of Λ, i.e. the algebra $\Lambda(K)$ obtained by dividing Λ by the graded ideal $I(K)$ spanned by the oriented simplices of S not in K. And now it follows, just as easily, that

$$(C(K))^\wedge \cong C(K^\wedge).$$

(D) STAR ALGEBRAS.

The definitions of (C) are in fact the case $G = 1$ of more general definitions valid for *any* group G. However, since the only other case which interests us at the moment is $G = \mathbb{Z}_2$, we will introduce distinguishing notation only for this case as follows:

We fix some total order on the N positive vertices of the octahedral sphere U.

This induces, *cf.* (A), a total order on each simplex of U. The vector space of all linear combinations of these totally ordered simplices of U will be denoted $\mathcal{L}(U)$ and its elements called the *ordered chains* of U. The dual vector space $L(U)$ consists of *ordered cochains* , i.e. functions defined on the set of ordered simplices of U. Since the conjugation of U preserves the total order of each simplex of U it determines a linear automorphism in each of these vector spaces, which will also be called *conjugation*.

To see the connection with (C), note that by associating to each ordered simplex of U the corresponding orientation, one gets a linear isomorphism of $\mathcal{L}(U)$ with the vector space $\mathcal{C}(U)$ of oriented chains of U, and likewise a linear isomorphism of the vector space of oriented cochains $C(U)$ with $L(U)$.

¿From now on when we refer to a simplex of U it will be understood that its vertices are totally ordered as above. Thus U is a *canonical basis* of both $\mathcal{L}(U)$ and $L(U)$ and provides us with a $\mathbb{Z}_2$-vector space identification $\mathcal{L}(U) = L(U)$.

We now identify this vector space further with the underlying vector space of the *free star algebra* Ω generated by our $2N$ signed vertices. Using $*$ to denote its product, this algebra is defined to be the associative $\mathbb{F}$-algebra with unity, generated by the signed vertices, subject to the relations

$$v * w = -w * v,\ v * \overline{w} = -w * \overline{v},\ \overline{v} * w = -\overline{w} * v,\ \overline{v} * \overline{w} = -\overline{w} * \overline{v},$$

for all positive vertices v and w.

Note that conjugation is an algebra automorphism of Ω, and that, unlike an exterior algebra, Ω is not (signed) commutative.

We now *grade* this algebra by *types* by proclaiming all nonzero elements contained in the vector space spanned by the (ordered as above nonempty) simplices of U of type α, to be of 'degree' α. If this degree α vector space direct summand of Ω is denoted Ω_α then one has

$$\Omega_\alpha * \Omega_\beta \subseteq \Omega_{\alpha*\beta}.$$

In the above equation, the binary operation $*$ on types is the one given in (A). Once again we refer to an element of Ω as decomposable if it is a product of two elements of strictly lesser (with respect to $\preccurlyeq$ of (A)) degrees. In particular note that each positive dimensional simplex of U is decomposable.

Important for the 'star shifting' of the next section will be the fact that, because of the bilinearity of $*$, the relations

$$x * y = -y * x,\ x * \overline{y} = -y * \overline{x},\ \overline{x} * y = -\overline{y} * x,\ \overline{x} * \overline{y} = -\overline{y} * \overline{x},$$

are valid even when the letters x and y denote *any* elements of Ω_1, the subspace spanned by the positive vertices.

If V and W are vector subspaces of Ω then the space spanned by all $\mu * \nu$, $\mu \in V$, $\nu \in W$, will be denoted by $V * W$.

Using this we associate, to each graded vector subspace V of Ω the graded $\mathbb{Z}_2$-vector subspace $V_* = V * \overline{V} + \overline{V} * V$. If $V = \sum_\alpha V_\alpha$, then the degree α summand $(V_*)_\alpha$ of V_* equals the (not necessarily direct) sum of the subspaces $V_\beta * \overline{V_\gamma}$ and $\overline{V_\delta} * V_\epsilon$ as (β, γ) and (δ, ϵ) run over all ordered pairs of types such that $\beta * \overline{\gamma} = \alpha = \overline{\delta} * \epsilon$.

If V is a graded subalgebra or ideal, then so is V_*.

So to each quotient graded algebra $A = \Omega/I \cong \oplus_\alpha \Omega_\alpha / I_\alpha$, where $I = \sum_\alpha I_\alpha$ is a graded ideal, we can associate the quotient graded algebra $A^* = \Omega/I_*$.

If K is a subcomplex of U, then the ordered chains $\mathcal{L}(K)$ of K constitute a graded vector subspace of Ω, *viz.* that which is spanned by the (ordered as above) simplices of K. From this it follows at once that

$$(\mathcal{L}(K))_* = \mathcal{L}(K_*).$$

On the other hand, the ordered cochains $L(K)$ of K constitute a quotient algebra of Ω, *viz.* that obtained by dividing Ω by the graded ideal $I(K)$ spanned by the simplices of U not in K. So it follows now that

$$(L(K))^* \cong L(K^*)$$

Again we see that the ordered cochains are somewhat superior to ordered chains, as $L(K)$ inherits from Ω a quotient star algebra structure $\Omega(K) = \Omega/I(K)$, while $\mathcal{L}(K)$ is merely a vector space.

In this context note also that an aforementioned linear isomorphism $C(U) \to L(U)$, which is not an algebra isomorphism, induces, for a $K \subseteq U$, an algebra isomorphism $C(K) \to L(K)$, if and only if all vertices of K have the same sign.

This remark shows that the 'star shifting' operation, which will be defined in the next section, coincides with Kalai's 'exterior shifting', in case all vertices of $K \subset U$ have the same sign.

(E) STAR SHIFTING.

Let K be a subcomplex of U, whose vertices are ordered as before, and let $L(K) = (\Omega(K), *)$ be the quotient star algebra of ordered cochains of K with coefficients in F, defined as in (D).

Using the common vector space basis K, of $L(K)$, and of the dual vector space of chains $\mathcal{L}(K)$, we get a vector space identification $\mathcal{L}(K) = L(K)$. The resulting inner product on this vector space is denoted by $\langle\ ,\ \rangle$.

So $\langle \nu, \mu \rangle = 0$ for all distinct pairs of (ordered as in (A)) simplices ν, μ of this *canonical basis* K while $\langle \nu, \nu \rangle = 1\ \forall \nu \in K$.

Let $c_1, c_2, \dots$ be any N elements of F with all van der Monde determinants nonzero. Then the vector subspace $\Omega_1 \subset \Omega$ spanned by the positive vertices $v_1, v_2, \dots,$ can be equipped with the new (totally ordered) basis $x_1, x_2, \dots$ given by

$$\langle x_i, v_j \rangle = (c_i)^j$$

Consider now any *typed word* in the *letters* x_i, i.e. an ordinary word (finite sequence) with possibly some of the letters overlined. We assign to each such word a *type* determined by this overlining. All typed words of a fixed type α will be *lexicographically ordered* as ordinary words (i.e. ignoring all overlining).

By taking in order the star product of its letters, or of their conjugates if overlined, such a typed word determines an element of the same type of $\Omega(K)$. Clearly such elements span the vector space $\Omega(K)$. Note further that words with some letter repeated (even with a change of its overlining) give zero, and permutations of the letters (with the overlining staying put at the same spots of the sequence) give elements differing at most upto sign.

So this spanning set is definitely not a vector space basis of $\Omega(K)$. To obtain such a one, we now 'seive' it as follows: we delete any typed word which determines an element of $\Omega(K)$ depending linearly on elements determined by lexicographically preceding words of that type. This results in a graded (by type) vector space basis of $\Omega(K)$ which will be denoted $\Delta_{\mathbf{c}}(K)$, or simply $\Delta(K)$.

Note that the letters in each typed word of $\Delta(K)$ are necessarily strictly increasing. So, if U_X denotes the octahedral sphere on the union X of the N *letter-pairs* $\{x, \overline{x}\}$, then $\Delta(K)$ identifies as in (A) with a simplicial subset of U_X.

Furthermore, if K is preserved by conjugation, then clearly so is $\Delta(K)$.

$\Delta_{\mathbf{c}}(K)$ *is closed under inclusion.*

Together with the preceding remark, it shows thus that the *star shifting* operation

$$K \rightsquigarrow \Delta_{\mathbf{c}}(K),$$

associates to each (*resp.* (free) $\mathbb{Z}_2$-) subcomplex K of U_Y a (*resp.* (free) $\mathbb{Z}_2$-) subcomplex $\Delta_{\mathbf{c}}(K)$ of U_X.

Proof. To see this note that if a decomposable determined by a typed word is a linear combination of lexicographically preceding decomposables of the same type, then a longer decomposable will also be such, because a new linear dependency results from the old when the new letters of the bigger decomposable are inserted throughout the dependency at the same places. *q.e.d.*

If the field $\mathbf{F}$ is big over its characteristic subfield $\mathbf{F}_p$, then one obviously has field elements c_i with van der Monde determinants nonzero. In fact we can now even choose them to be algebraically independent over $\mathbf{F}_p$. In this case, we will call $\Delta_{\mathbf{c}}(K)$ a *generic basis* of $L(K)$. We have seen that, like the canonical basis K, $\Delta(K)$ is always a simplicial complex. We now show that, in this case, it is of a very special kind.

A generic $\Delta(K)$ *is* type-shifted, *i.e. is closed with respect to the product partial orders on typed words of the same type.*

Proof. Let compatibly typed words θ and σ, with letters strictly increasing, be such that the letters $x_{\theta,i}$ of θ are respectively less than or equal to the letters $x_{\sigma,i}$ of σ. Because of the assumed algebraic independence of the c_k's over $\mathbf{F}_p$, there is a field automorphism of $\mathbf{F}$ over $\mathbf{F}_p$ which images each $c_{\theta,i}$ to $c_{\sigma,i}$, and the c_k's other than these $c_{\theta,i}$'s, in order, to the c_k's other than these $c_{\sigma,i}$'s. This determines a type preserving (and also equivariant, in case K is preserved by the conjugation) F_p-algebra isomorphism $\Omega(K) \to \Omega(K)$.[12] Under it any linear dependency of θ, in terms of compatibly typed lexicographically preceding words, will image to a similar linear dependency of σ. *q.e.d.*

(F) COBOUNDARY.

Let $\mathbf{T} = \mathbf{T}_Y$ denote the *tensor algebra,* i.e. the free associative algebra, generated by the set $Y = Y_+ \cup Y_-$ of our $2N$ (positive or negative) vertices. It has an F-vector space basis consisting of all finite sequences s (possibly with repetitions) of vertices, and the tensor product $s.t$, of two such sequences s and t, is merely their juxtaposition.

There is an action of $\mathbb{Z}_2$ on $\mathbf{T}$ given via the algebra automorphism $s \mapsto \overline{s}$.

Each sequence $s = v_0.v_1.\dots.v_{|s-1|}$ of positive or negative vertices, factors uniquely as a product of maximal subsequences s_i, whose vertices are, alternately, all positive, or all negative. Thus s has a well defined *type* indicating, in order, the signs and lengths $|s_i|$ of these subsequences. We use this to *grade* $\mathbf{T}$ by type, analogously to its quotient Ω considered before.

Using the above graded basis of $\mathbf{T}$ we now define a linear map, the (usual) *coboundary* $\delta : \mathbf{T} \to \mathbf{T}$, by

$$\delta(v_0.v_1.\dots.v_{|s-1|}) = \sum_{r=0}^{|s|}(-1)^r(\delta_{(r,+)} + \delta_{(r,-)})(v_0.v_1.\dots.v_{|s-1|}),$$

where the homogenous (with respect to type-grading) maps $\delta_{(r,\pm)}$ are defined, on all types having total length r or more, by

$$\delta_{(r,\pm)}(v_0.v_1.\dots.v_{|s-1|}) = (\dots.v_{r-1}.(\sum_{Y_\pm} v).v_r.\dots).$$

Here, and below, it is understood that all our coboundaries are *augmented,* i.e. their action on the scalars F is given by mapping $1 \in \mathsf{F}$ to the element determined by the sum of all the vertices.

By linearity it follows that this *defining formula*

$$\delta(x_0.x_1.\dots.x_{|s-1|}) = \sum_{r=0}^{|s|}(-1)^r(\delta_{(r,+)} + \delta_{(r,-)})(x_0.x_1.\dots.x_{|s-1|}),$$

[12] Thus each generic c determines a homomorphic image of the Nth symmetric group within the group of all graded F_p-algebra automorphisms of $\Omega(K)$.

where

$$\delta_{(r,\pm)}(x_0.x_1.\dots.x_{|s-1|}) = (\dots.x_{r-1}.(\sum_{Y_\pm} v).x_r.\dots),$$

remains valid for *any* word $x_0.x_1.\dots.x_{|s-1|}$ whose letters are elements of $\mathbf{T}_1$ or their conjugates.

For each $s \leq r$, one has the (four) commutation relations,

$$\delta_{(s,\pm)} \circ \delta_{(r,\pm or \mp)} = \delta_{(r+1,\pm or \mp)} \circ \delta_{(s,\pm)},$$

which imply that the coboundary $\delta : \mathbf{T} \to \mathbf{T}$ is of *order two,* i.e. is such that $\delta \circ \delta = 0$.

(G) CYCLIC COBOUNDARY.

We now define a 'suitable coboundary' $L(K) \to L(K)$ well-behaved for all field characteristics.

Before doing this we remark that though the (usual) coboundary δ : $\mathbf{T} \to \mathbf{T}$ of (F) *does* induce a map in Ω, and still further in $L(K) = \Omega/I(K)$, for all $K \subseteq U$, this ***induced coboundary*** δ (which obeys the defining formula of (F) with 'stars' instead of 'dots') is *not* the one which should be used when the coefficients F have a nonzero characteristic.

This is already quite clear even for the (sign) 'commutative case' $K \subseteq S \subset U$ (when the star algebra $L(K)$ is an exterior algebra) because this induced map amounts to *first* multiplying each simplex by one more than its cardinality, and only then taking the usual ***reduced simplicial coboundary.***

In this commutative case, the replacement of the induced coboundary, by this reduced coboundary, can be viewed as follows:

Now the type-grading coincides with that by (one more than) dimension, and furthermore, *all* the nonzero summands $(-1)^r\delta_{(r,+)}$'s of the induced map have the same degree +1.

In fact, even more is true: because of commutativity, all these summands $(-1)^r\delta_{(r,+)} : L(K) \to L(K)$ are the *same* . The reduced simplicial coboundary $L(K) \to L(K)$ is obtained by choosing *any one* of these summands instead of the induced coboundary δ.

By choosing $r = 0$, we see that on any $\sigma \in L(K)$, $K \subseteq S \subset U$,the action of this reduced map is given by the coboundary formula

$$\sigma \mapsto \Upsilon \wedge \sigma,$$

where $\Upsilon \in L_1(K)$ denotes the *sum of the positive vertices.*

We now return to the general case $K \subseteq U$, and define the *cyclic coboundary* $\delta_{cyc} : L(K) \to L(K)$, by again throwing out[13] some of the summands $(-1)^r \delta_{(r,\pm)}$ from the induced coboundary δ of $L(K)$.

More precisely, δ_{cyc} is the linear map, commuting with conjugation, satisfying the following *coboundary formula* for any $\sigma \in L(K)$ which is slot factorizable (as in (A) with the last slot possibly empty):

$$\delta_{cyc}(\sigma) = \delta_{cyc}(\sigma_1 * \overline{\sigma_2} * \cdots * \sigma_{2s-1} * \overline{\sigma_{2s}}) =$$

$$\sum_{t=1}^{s}[(-1)^{\cdots+|\sigma_{2(t-1)}|}(\cdots * (\Upsilon * \sigma_{2t-1}) * \overline{\sigma_{2t}} * \cdots)+$$

$$(-1)^{\cdots+|\sigma_{2t-1}|}(\cdots * \sigma_{2t-1} * (\overline{\Upsilon} * \overline{\sigma_{2t}}) * \cdots)].$$

Again the verification of $\delta_{cyc} \circ \delta_{cyc} = 0$ is straightforward.

Note that $\delta_{cyc} : L(K) \to L(K)$ coincides with the reduced simplicial coboundary, and the above coboundary formula reduces to the one given before, for the commutative case $K \subseteq S$.[14]

We now define the *cyclic subcomplex* of $(L(K), \delta_{cyc})$ as follows:
....

(H) COBOUNDARY DEFORMATIONS.
In this section we abbreviate δ_{cyc} to δ since only cyclic coboundary is used, but note that the definitions and the propositions have obvious analogues for induced coboundary, and also for the 'type-reduced coboundary' considered later in (J).

It is easily seen that a $\mathbb{Z}_2$-graded algebra isomorphism $c : \Omega \to \Omega$ is uniquely determined by the restricted linear isomorphism c of the subspace Ω_1 spanned by the positive vertices.

We will call such a $c : \Omega \to \Omega$ a *simplicial automorphism* if it preserves *all* ideals of the form $I(K)$, K a subcomplex of U, and so induces a

[13] The choices of the particular summands which will be thrown out was motivated, besides the result of (I), by the *cyclic cohomology* which Connes has used in characteristic zero.

[14] Also note the ease with which the definition of δ_{cyc} will generalize when we consider groups G other than $\mathbb{Z}_2$.

graded algebra automorphism $c : L(K) \to L(K)$ for all $K \subseteq U$. Such a quotient (star) algebra automorphism will also be called simplicial.

For any simplicial algebra automorphism c of $L(K)$ one has

$$c \circ \delta = \delta_{[c]} \circ c,$$

where $\delta_{[c]} : L(K) \to L(K)$ obeys the coboundary formula of (G) *provided the sum Υ of the positive vertices is replaced by their linear combination $x = \sum_{Y_+} c(v) \cdot v$.*

Proof. This follows at once from the coboundary formula, because an automorphism c of Ω is clearly simplicial if and only if

$$\overset{\circ}{c} = diag(c_1, c_2, \ldots),$$

with respect to the canonical basis of Ω. In other words c multiplies each vertex v_i (and so also its conjugate $\overline{v_i}$) by c_i, and thus any sequence s of positive or negative vertices, with the corresponding monomial in the c_i's. Note also that because c is one-one all these monomials in the c_i's are necessarily nonzero. *q.e.d.*

In fact for *any* (star) graded algebra automorphism $c : L(K) \to L(K)$ (simplicial or not), there is a corresponding *deformation* $\delta_{[c]} = c \circ \delta \circ c^{-1}$ of the cyclic coboundary δ, which also is obviously of order two, and which obeys a coboundary formula.

For example, there is a $\mathbb{Z}_2$-graded algebra automorphism

$$\geq : \Omega_X \to \Omega_X,$$

which maps the first letter $x = x_1$ to the sum of the positive letters, and so its conjugate $\overline{x}$ to the sum of the negative letters, and keeps all other letters fixed.

If a subcomplex $\Delta \subseteq U_X$ is type-shifted (see (E)) it is easily seen that its ideal $I(\Delta)$ is preserved by this automorphism $\geq$. Thus, for any such subcomplex, there is an induced algebra automorphism $\geq : L(\Delta) \to L(\Delta)$.

For any type-shifted subcomplex $\Delta \subseteq U_X$, the deformation $\delta_{[\geq]}$ of the coboundary obeys the coboundary formula of (G) *provided the sum Υ of the positive letters of U_X is replaced by the least positive letter x.*

Proof. This follows at once from the coboundary formula and the definition of $\geq$. *q.e.d.*

Note that the formula for $\delta_{[\geq]}$, given by the above proposition, makes sense even when Δ is *not* type-shifted, so using it we extend the definition of $\delta_{[\geq]} : L(\Delta) \to L(\Delta)$ to *all* $\Delta \subseteq U_X$.

(I) SEIVING ISOMORPHISM.

In this section too we abbreviate δ_{cyc} to δ since only cyclic coboundary is used.

Note that the cyclic coboundary $\delta : L(K) \to L(K)$, $K \subseteq U$, is not homogenous with respect to the (type) grading of L(K), but is ofcourse homogenous of degree +1 with respect to the *coarser grading* of L(K) by (one more than) dimension.

The same remark applies to the cyclic coboundary δ of $L(\Delta)$, $\Delta \subseteq U_X$, and to the maps $\delta_{[c]}$ and $\delta_{[\geq]}$ of (H).

To understand the common behaviour of these maps with respect to the (finer type) grading, we need to look at their formulae more closely.

We do this using the *notations σ_K and σ_Δ* to denote the elements of $L(K)$, $K \subseteq U$, and $L(\Delta)$, $\Delta \subseteq U_X$, respectively, determined by any (totally ordered as in (A)) word σ of U_X.

If $\sigma \in U_X$ is of type α and of length $2s$, with the first term of α say positive, then the formulae for $\delta_{[c]}(\sigma_K)$ and $\delta_{[\geq]}(\sigma_\Delta)$ run

$$\delta_{[c]}(\sigma_K) = \sum_{t=1}^{2s}(x_K *_t\sigma_K) \text{ and } \delta_{[\geq]}(\sigma_\Delta) = \sum_{t=1}^{2s}(x_\Delta *_t\sigma_\Delta),$$

where ${}_t\sigma \in U_X$ are some distinct *mutants* $\tilde{\sigma}$ of the word σ, i.e. words obtained from σ merely by a change of overlining. We will denote the corresponding distinct (but equicardinal) mutant types $\tilde{\alpha}$ by $\{{}_1\alpha, {}_2\alpha, \ldots, {}_{2s}\alpha\}$. Note that here ${}_1\sigma = \sigma$, and so ${}_1\alpha = \alpha$.

If the first term of α is negative all the x_K's and x_Δ's get replaced by $\overline{x}_K$'s and $\overline{x}_\Delta$'s[15], and analogous formulae hold also for the δ's.

We now equip the (not necessarily generic) star-shifting operation $K \rightsquigarrow \Delta_c(K)$, where $K \subseteq U$ and $\Delta_c(K) = \Delta \subseteq U_X$, of (E) with a linear *seiving isomorphism*

$$\blacktriangle = \blacktriangle_c^K : L(K) \to L(\Delta_c(K)),$$

[15] Because of this we can, and will, w.l.o.g. only work out the case when the first term of α is positive

which too will be homogenous (of degree 0) *only* with respect to the coarser gradings of the two sides.

With respect to the finer type grading this map will be the direct sum

$$\blacktriangle = \sum_{\alpha} \blacktriangle_{\alpha} \quad \text{where} \quad \blacktriangle_{\alpha} = \sum \blacktriangle_{\alpha\beta}\,,$$

where β runs over all mutants $\tilde{\alpha}$ of the type α, and the homogenous linear maps $\blacktriangle_{\alpha\beta}$ are as follows.

Definition of $\blacktriangle_{\alpha\beta} : L_{\alpha}(K) \to L_{\beta}(\Delta)$, $\Delta = \Delta_{\mathbf{c}}(K)$:

Note first that as σ runs over the words of $\Delta = \Delta_{\mathbf{c}}(K)$, the elements σ_K and σ_{Δ}, provide us with a basis, and *the* canonical basis, of $L(K)$ and $L(\Delta)$, respectively.

For any $\sigma \in \Delta$, we put

$$\text{(a)} \qquad \blacktriangle_{\alpha\beta}(\sigma_K) = (\tilde{\sigma})_{\Delta}$$

unless there is a lexicographic dependency $x_K * (\tilde{\sigma})_K = \sum_{\theta<\sigma} q_{\theta}(x_K * (\tilde{\theta})_K)$, with $x(\tilde{\theta}) \in \Delta$, when we put

$$\text{(b)} \qquad \blacktriangle_{\alpha\beta}(\sigma_K) = (\tilde{\sigma})_{\Delta} + \sum_{\theta<\sigma} q_{\theta}(\tilde{\theta})_{\Delta}.$$

The linear map $\blacktriangle : L(K) \to L(\Delta)$ *just defined is indeed an isomorphism, and one has*

$$\blacktriangle \circ \delta_{[\mathbf{c}]} = \delta_{[\geq]} \circ \blacktriangle,$$

where $\delta_{[c]} : L(K) \to L(K)$ *and* $\delta_{[\geq]} : L(\Delta) \to L(\Delta)$ *are the maps defined in* (H).

Proof. Since the matrix of $\blacktriangle_{\alpha\alpha}$, with respect to the bases used in the above definition, is lower-triangular with ones on the diagonal, $\blacktriangle$ is one-one.

For any type α (with again say the first term positive) the types $\beta = {}_t\alpha$, $t \geq 1$, and so necessarily also the types $\gamma = 1 * {}_t\alpha, t \geq 1$, are all *distinct*.

So it suffices only to check, for all words $\sigma \in \Delta$ with first term positive, that

$$\blacktriangle_{\gamma\gamma}(x_K * (\tilde{\sigma})_K) = x_{\Delta} * \blacktriangle_{\alpha\beta}(\sigma_K),$$

which follows because of the 'correction terms' which we took care to add in the case (b) of the above definition. *q.e.d.*

Generically, the map

$$(\geq)^{-1} \circ \blacktriangle_{\mathbf{c}}^{K} \circ c : L(K) \to L(\Delta_{\mathbf{c}}(K)),$$

where c and $\geq$ denote the graded algebra automorphisms of (H), *commutes with the cyclic coboundaries δ of $L(K)$ and $L(\Delta_{\mathbf{c}}(K))$*

Proof. This follows from the above result and (H) where we saw that $c \circ \delta = \delta_{[c]} \circ c$ always, while $\geq \circ \delta = \delta_{[\geq]} \circ \geq$ for $\Delta = \Delta_{\mathbf{c}}(K)$ type-shifted, which by (E) is the case generically. *q.e.d.*

Two more properties of $\blacktriangle$: (1) functoriality (2) maps cyclic subcomplex to cyclic subcomplex, ...

(J) TYPE-REDUCED COBOUNDARY.

In (G) we defined the cyclic coboundary δ_{cyc} by throwing away some summands from the coboundary δ induced in $L(K)$ by the natural quotient map $\mathbf{T}(K) \to L(K)$.

We will now consider yet another coboundary, still suitable for working over any field characteristic, which involves the throwing away of a far lesser number of summands.

The defining formula of $\delta : \mathbf{T} \to \mathbf{T}$ shows that an element of $\mathbf{T}_\alpha$ gets mapped by δ into a sum of elements of types

$$\gamma = 1 * \alpha,\ \bar{1} * \alpha, \ldots,\ 1 *_{(2t-1)}\alpha,\ \bar{1} *_{2t}\alpha, \ldots,$$

where, for $s \geq 1$, the types ${}_s\alpha$ are distinct mutants of α.

Next we note, because of the limited (within slots) commutativity of $\Omega(K)$, that all summands $(-1)^r \delta_{(r,\pm)}$ of δ, which map $\mathbf{T}_\alpha$ to the same $\mathbf{T}_\gamma$, are in fact themselves same.

We define the *type-reduced coboundary* $\delta_{red} : L(K) \to L(K)$ by keeping, for each γ, just one of these summands.

So δ_{red} has the *coboundary formula*

$$\delta_{red}(\sigma) = (\Upsilon + \overline{\Upsilon}) * \sigma + (\cdots + \Upsilon *_{(2t-1)}\sigma + \overline{\Upsilon} *_{2t}\sigma + \cdots),$$

where, for $s \geq 1$, the mutants ${}_s\sigma$ of σ are in the distinct types ${}_s\alpha$ mentioned above.

The verification of $\delta_{red} \circ \delta_{red} = 0$ is straightforward.

For the group $G = \mathbb{Z}_2$ with which we are working, an *equivariant cochain* is one which is either symmetric or skewsymmetric. These form the subspaces

$$L_\pm(K) = \{\lambda \in L(K) : \overline{\lambda} = \pm\lambda\}$$

of $L(K)$.

If K is not preserved by conjugation, these subspaces can be very small indeed: e.g. if K has all vertices positive, then there is no nonzero equivariant cochain.

On the other hand, if K is preserved by the conjugation, and if the field characteristic is not 2, then $L(K) = L_+(K) \oplus L_-(K)$. And, if now, the field characteristic is 2, then the two subspaces $L_\pm(K)$ coalesce into a single subspace of $L(K)$, of half its dimension.

Note also that all maps defined in (H) and (I), and all our coboundaries — induced, cyclic, or type-reduced — commute with conjugation, and thus preserve the equivariance (i.e. symmetry or skewsymmetry) of a cochain.

The generic equivariant seiving isomorphisms

$$\blacktriangle_{\mathbf{c}} : L_\pm(K) \to L_\pm(\Delta(K))$$

commute with the deformations $\delta_{[\mathbf{c}]}$ and $\delta_{[\geq]}$ of the type-reduced coboundaries, $\delta = \delta_{red}$, of $L(K)$ and $L(\Delta)$.

Proof. A basis of $L_\pm(K)$ is provided by elements of the type $\sigma \pm \overline{\sigma}$, where we can assume that the first term in the type α of σ is *positive.*

The coboundary formulae for the aforementioned deformations are like the one given above for δ_{red} excepting that $\Upsilon, \overline{\Upsilon}$ get replaced by $x_K, \overline{x}_K$ and $x_\Delta, \overline{x}_\Delta$ respectively.

We saw in (I) that

$$\blacktriangle_{\gamma\gamma}(x_K * \sigma_K) = x_\Delta * \blacktriangle_{\alpha\alpha}(\sigma_K) \text{ and } \blacktriangle_{\overline{\gamma\gamma}}(\overline{x}_K * \overline{\sigma}_K) = \overline{x}_\Delta * \blacktriangle_{\alpha\alpha}(\overline{\sigma}_K).$$

In addition, since $\blacktriangle$ commutes with conjugation, one has

$$\blacktriangle_{\overline{\nu\nu}}(\overline{\lambda}) = \overline{\blacktriangle_{\nu\nu}(\lambda)}$$

for all $\lambda \in L_\nu(K)$.

Using these it follows that

$$\blacktriangle((x_K + \overline{x}_K) * (\sigma_K \pm \overline{\sigma}_K) = (x_\Delta + \overline{x}_\Delta) * \blacktriangle(\sigma_K \pm \overline{\sigma}_K).$$

Because of the distinctness of the remaining mutants ${}_s\alpha$, $s \geq 1$, the commutativity of the remaining terms of the coboundary formulae with $\blacktriangle$ is even easier and follows as in (I). *q.e.d.*

Of course, by (I), the same result is true also for the cyclic coboundary.

Note that, for the case $K = \overline{K}$, and field characteristic not equal to 2, we have now checked, over *all* of $L(K)$, the commutativity of ▲ even with type-reduced coboundary. But, for characteristic two, we have checked this *only* over the half-dimensional equivariant subcomplex $L_{\pm}(K)$.

Likewise, when $K = \overline{K}$ and the field characteristic is zero, the (now useful) induced coboundary commutes with ▲ over all of $L(K)$.

(K) UNIFORM COMPLEXES.

A set $\mathfrak{t}$ of types will be called an *ideal of types* if, for any simplex of U having its type in $\mathfrak{t}$, all faces of the simplex also have their types in $\mathfrak{t}$.

Given such a $\mathfrak{t}$, a simplicial complex $K \subseteq U$ will be called $\mathfrak{t}$-*uniform*, iff equicardinal (and ordered as in (A)) simplices having their types in $\mathfrak{t}$, and differing from each other only in their overlining, are either all in K or all outside K.

On the other hand, the $\mathfrak{t}$-*skeleton* of a $K \subseteq U$ will be the subcomplex consisting of all simplices of K having their types in $\mathfrak{t}$.

For each $K \subseteq U$, we have *the smallest* $\mathfrak{t}$-*uniform simplicial complex* containing K, as well as *the largest* $\mathfrak{t}$-*uniform complex* contained in the $\mathfrak{t}$-skeleton of K.

The ordered chain spaces $\mathcal{L}$ of these complexes can be calculated directly from the chain space $\mathcal{L}(K)$ of the given K, and the ideal $\mathfrak{t}$, without using any thing else.

Star shifting preserves $\mathfrak{t}$-*uniformity.*

Proof. This follows easily from the definition since the 'change-of-overlining' (within $\mathfrak{t}$), which is given to be an isomorphism commutes with the seiving process. *q.e.d.*

So, for any $K \subseteq U$, $\Delta(K)$ is contained between the star shifts of the aforementioned two $\mathfrak{t}$-uniform complexes, between which K is nested.

For applications to embeddability we need to look at the *deleted join*

$$D = K_* = \{\sigma \cup \overline{\theta} : \sigma \cap \theta = \emptyset\},$$

the free $\mathbb{Z}_2$-simplicial subcomplex of U associated to a $K \subseteq S$.

For any ideal $\mathfrak{t}$ of types the largest $\mathfrak{t}$-uniform subcomplex of D will be called the $\mathfrak{t}$-*uniform deleted join* of K.

Of importance will be the ideal $\mathfrak{t}$ containing all types below the conjugate types $n+1$, $\overline{n+1}$ and $\overline{n+1}$, $n+1$. And, more generally all types below s_1, $\overline{s_2}$, ... and $\overline{s_1}$, s_2, ... Here $n = dim K$, and $s_1 + s_2 + \cdots$.

The star shift of the $\mathfrak{t}$*-uniform deleted join of a* $K \subseteq S$*, contains the* $\mathfrak{t}$*-uniform deleted join of its exterior shift* $\Delta(K) \subseteq S_X$.

Before turning to the proof we note that, for the aforementioned choices of $\mathfrak{t}$, this result implies that a Kuratowski n-complex T is contained in the corresponding type of $\Delta(K_*)$, only if its deleted join is also contained.

comment

(K) DELETED JOINS.

There is a simplicial surjection $\phi : U \to S$, *viz.* that which maps any vertex-pair $\{v, \overline{v}\}$ to the positive vertex v.

Thus each subcomplex K of U surjects to a subcomplex $\phi(K)$ of S.

Next, consider any *order ideal* of types $\mathfrak{t}$, i.e. if a simplex of U has a type from $\mathfrak{t}$, then any face should also have a type from $\mathfrak{t}$. Let us denote by $K_{\mathfrak{t}}$ the subcomplex of a $K \subset U$ formed by all simplices of K having types in $\mathfrak{T}$.

For any $K \subseteq U$, consider the intersections of the simplicial sets $\phi(K_\alpha)$, where K_α denotes the subset of K consisting of all simplices of type α, and α runs over all equicardinal types in $\mathfrak{t}$. These constitute a subcomplex of $\phi(K_{\mathfrak{t}})$. Its pre-image under ϕ will be denoted $K_{[\mathfrak{t}]}$, and said to be the *uniformization* of the complex K over the order ideal of types $\mathfrak{t}$.

Note that such a uniform complex $D = K_{[\mathfrak{t}]}$ is determined entirely by $\phi(D)$ and the types ideal $\mathfrak{t}$.

Let us now temporarily reserve the letter K *to denote a simplicial complex with all vertices positive.*

For any such $K \subseteq S$ we have its *deleted join* $D = K_*$, the free $\mathbb{Z}_2$-simplicial complex consisting of all simplices of the type $\{(\sigma \cup \overline{\theta}\}$, where σ and θ are disjoint simplices of K.

Note that $\phi(D) \subseteq S$ consists of all simplices which can be partitioned into two disjoint simplices of K, so of course $\phi(D)$ is determined by K. But note that the knowledge of $\phi(D)$, and all the types occuring in D does not determine D.

However for any uniformization $D_{\mathfrak{t}} \subset U$, it is so: K and the ideal $\mathfrak{t}$ determine this projection $\phi(D_{\mathfrak{t}})$, which together with $\mathfrak{t}$ determines the uniformized subcomplex.

By a *uniform deleted join* $E \subseteq K_*$ of K we will mean some such uniformization of its deleted join $D = K_*$.

A uniform deleted join E of K star shifts to a complex $\Delta(E)$ containing the corresponding uniform deleted join of $\Delta(K)$.

Before turning to the proof of the above proposition let us see what it implies.

Consider first that $dimK = n$ and that K in fact contains two disjoint n-simplices. Let $\mathfrak{t}$ be all types below the types $n+1$, $\overline{n+1}$ and $\overline{n+1}$, $n+1$. If $\Delta(K)$ contains the n-skeleton of a $2n+2$-simplex, then $\Delta(K_*)$ will contain its uniform deleted join over these types, which is the antipodal $(2n+1)$-sphere of Flores.

But were we just asserting that some reducible Kuratowski n-complex is in $\Delta(K)$, then this uniform deleted join is much smaller than a sphere.

However if $\Delta(K_*)$ were to contain such a reducible complex in the corect type, then for $\mathfrak{t}$ now the types below this type and its conjugate, the uniform deleted join would coincide with the usual one for a reducible Kuratowski complex.

endcomment

(M) SINGULAR COCHAINS.

For any subcomplex $K \subseteq U$, the process of dividing $\mathbf{T}$, the free associative algebra of (F), by the ideal determined by words not supported on K, gives a quotient tensor algebra $\mathbf{T}(K)$, which we will equip with the coboundary induced from that of $\mathbf{T}$.

In this very large algebra, of which $L(K)$ is a finite-dimensional quotient, the induced coboundary, is *quite* suitable in any field characteristic.

In fact the words supported on simplices of K determine a canonical basis of $\mathbf{T}(K)$. And further, any such sequence of length t determines, and is determined by, a simplicial (so continuous) map from a mutant of the standard closed simplex $[1, 2, \ldots, t]$ to K.

As such, $(\mathbf{T}(K), \delta)$, which will be referred to as the *semi-simplicial cochain complex* of K, identifies naturally with a sub cochain complex of the (still larger) *singular typed-cochain complex* of K, which has as basis *all* continuous maps from such *typed standard closed simplices* to K. It can be checked, using usual methods, that the cohomology of this singular typed-cochain complex is the usual cohomology of K.

The above inclusion, or for that matter, a similar inclusion of *equivariant* (i.e., for $\mathbf{Z}_2$, symmetric or skewsymmetric) semi-simplicial cochains within all singular cochains of the same equivariance, all induce an isomorphism in cohomology.

We will henceforth think of such an isomorphism as an identification, and accordingly, refer to the cohomology of $\mathbf{T}(K)$ as the cohomology of

K, and to those of the subcomplexes

$$\mathbf{T}_{\pm} = \{\lambda : \overline{\lambda} = \pm\lambda\},$$

as the *equivariant cohomologies* of K.

Thanks to the total order with which the vertices of U are equipped, there is also a natural injection

$$L(K) \to \mathbf{T}(K),$$

defined as follows.

§9. Simplicial spheres.

Following Kalai, we will now show how Theorem 6 of §8 (concerning the sharp Heawood inequalities), is in fact equivalent to McMullen's g-conjecture for simplicial spheres.[16]

Except for the use of this theorem, and the fact (see §7) that exterior shifting of simplicial complexes preserves the Cohen-Macaulay property, the arguments of this short section are purely combinatorial.

UPPER BOUND THEOREM. *For any simplicial sphere K there is an order ideal of monomials whose face polynomial is $h_K(z)$.*

Proof. Since $h_K(z) = h_\Delta(z)$ we can consider the shifted complex Δ.

We will denote by d a number one more than the dimension of K, so d is the cardinality of any maximal simplex of Δ.

We assert that h_k equals the number of maximal simplices of Δ which contain the first $d-k$ vertices, but not the next one.

This follows from the shelling interpretation of the h-vector of Δ given in §1 , because it is easy to shell the pure shifted complex Δ in such a way that this set $\mathbf{h}_k$ of top simplices corresponds to the shelling steps of type k.

Now for each such simplex, from the first vertex after the $(d-k)$th subtract $d-k-1$, from the next vertex one more, and so on. This gives a monomial. Denote all such monomials by $\mathfrak{h}_k$.

The union of all these $\mathfrak{h}_k$'s is the required order ideal of monomials $\mathfrak{h}$. **q.e.d.**

[16] Previous to this, McMullen's conjecture was known *only* for simplicial convex polytopes, as a *consequence* of the hard Lefschetz theorem (see §15) which holds for their associated toric varieties.

McMULLEN'S CONJECTURE. *For any simplicial sphere K there is an order ideal of monomials whose face polynomial is $g_K(z)$.*

Proof. We continue using the notations of the last proof.

With $k \leq \frac{d}{2}$ a simplex of $\mathbf{h}_{d-k}$ must contain the kth vertex, but not the next one, and then all upto and including the $(d-k+1)$th. This follows because otherwise (*cf.* proof of Theorem 2(f) of §3) it will dominate a $\sigma_t^t \cdot \sigma_{s-1}^{2s}$ with $t+2s = m+1$, which is forbidden by Theorem 6 of §8.

So deleting the $(d-k+1)$th vertex and replacing it by the $(k+1)$th constitutes an injection of the set $\mathbf{h}_{d-k}$ into the set $\mathbf{h}_k$.

But by the *functional equation* of §1 these 2 sets have the same cardinality. So the above injection is in fact a bijection.

The corresponding bijection $\mathfrak{h}_{d-k} \to \mathfrak{h}_k$ of monomial sets is multiplication by the $(d-2k)$th power of the first letter x_1.

Thus, in the order ideal $\mathfrak{h}$ of monomials, multiplication by x_1 is an injection on monomials of degree $\leq \frac{d}{2}$.

So the degree k monomials $\mathfrak{g}_k$ which do not lie in the image of such an injection, have cardinality $h_k - h_{k-1}$.

The union $\mathfrak{g}$ of these $\mathfrak{g}_k$'s is an order ideal because it coincides with the set of all monomials of $\mathfrak{h}$ not containing the first and the last vertices, x_1 and x_{N-d}, occuring in $\mathfrak{h}$. **q.e.d.**

The numerical versions of these theorems follow from the numerical version of Macaulay's theorem given in §6.

For the case of simplicial spheres which occur as the boundaries of convex polytopes, we will see in §15 that the h_k's coincide with the Betti numbers of the associated *toric variety* . This will enable us to show that *the last result implies the hard Lefschetz theorem for these toric varieties.* All known proofs of this theorem are quite different and use extensive algebraic-geometrical machinery.

The analogous interplay between the Deligne-Weil theorem, and the hard Lefschetz theorems for varieties over finite fields, suggests strongly that there exists a further generalization of McMullen's conjecture in the form of an interesting combinatorial *Riemann hypothesis*. In this context note that, using just the analogy with the functional equation of the zeta function, we have already discussed some "Riemann hypotheses" in §2.

§10. Symmetric shifting.

This time let us use the *chains* $L_*(B)$, with coefficients from a big field F, to linearly shift the order ideal of monomials B. So we choose a generic totally ordered basis $x_1, x_2, \ldots$ for the F-vector space V spanned by the vertices of B. Then the monomials in the x_i's span $L(B)$, and the subset of all lexicographically leading monomials of this F-vector space, is the required $\Delta_{\mathsf{F}}(B)$.

As before, it follows easily that $\Delta_{\mathsf{F}}(B)$ is an order ideal of monomials, and that, being generic, it is also shifted.

THEOREM 1. *There is a canonical linear chain isomorphism from* $L(\Delta_{\mathsf{F}}(B))$ *to* $L(B)$.

Likewise, there is also a linear cochain isomorphism, running in the opposite direction.

Proof. Since the details are similar to before we only sketch the construction. The first part

$$U : L(\Delta_{\mathsf{F}}(B)) \to L(\Delta_{\mathsf{F}}(B)),$$

is available because $\Delta = \Delta_{\mathsf{F}}(B)$ is shifted, and is an isomorphism converting the boundary operator ∂ to the 'local' boundary ∂_1 at the first letter x_1. This map U is identity on monomials containing x_1, while to any other monomial it adds all monomials obtainable by replacing an occurence of some letter by x_1.

Then there is an isomorphism

$$L : L(\Delta_{\mathsf{F}}(B)) \to L(B)$$

commuting with these differentiations ∂_1. The correction term suggests itself from the seiving interpretation of the shifting operation.

The last factor is the algebra isomorphism

$$D : L(B) \to L(B)$$

dividing each vertex v of B with the value of x_1 on v and thus converting the local boundary ∂_1 to ∂. **q.e.d.**

Thus the commutative semi-simplicial complexes B and $\Delta(B)$ have isomorphic (co)homology groups over F.

We will say that a commutative semi-simplicial complex B is *Cohen-Macaulay* if it obeys the condition given in §5 for the $B(K)$ of a Cohen-Macaulay simplicial complex K.

THEOREM 2. *An order ideal B of monomials is Cohen-Macaulay over* F *if and only if* $\Delta_{\mathsf{F}}(B)$ *is also Cohen-Macaulay.*

Proof. Two of the above maps (i.e. excluding U) are available even with restriction on vertex repetition.

We use this, and a spectral sequence argument, for one direction. (Other?) **q.e.d.**

Since our interest resides mainly in the order ideal of monomials $B(K)$ associated to a simplicial complex we now study this construction.

THEOREM 3. PROPERTIES OF $K \rightsquigarrow B(K)$.

(a) *The number $h_k(B)$ of degree k monomials of $B(K)$ are enumerated by the power series*

$$\sum_{k=0}^{\infty} h_k(B) \cdot z^k = \sum_{k=0}^{\infty}(\sum_{r=0}^{k} f_r(K)\binom{k-1}{r}) \cdot z^k.$$

Proof. Follows because the number of degree k monomials supported on an r simplex is $\binom{k-1}{r}$ *q.e.d.*

REMARKS. (1) Thus a necessary condition for B to be a $B(K)$ is that its face (or Poincaré / or Hilbert) series be of *finite type* , i.e.

$$\text{(I)} \qquad \sum_{k=0}^{\infty} h_k(B) \cdot z^k = \sum_{k=0}^{\infty}(\sum_{r=0}^{k} f_r\binom{k-1}{r} \cdot z^k,$$

for some finite sequence $\{1, f_0, f_1, \dots\}$ of positive integers.

However this is not a sufficient condition, e.g. the order ideal

...

(2) Using the fact that the face numbers of any order ideal obey Macaulay's conditions, it can be shown for this finite type case that these numbers f_k must obey Kruskal's conditions. However a more informative proof is the following.

(b) *If the Hilbert series of an order ideal of monomials B has the above finite type I, then there is a simplicial complex K for which $f_r(K) = f_r$ for all r.*

Proof. Since symmetric shifting preserves the Hilbert series, we can assume that B is shifted.

For shifted order ideals B we now define

$$B \rightsquigarrow K(B)$$

as follows.

One checks that B has precisely f_j monomials $x_{i_1} x_{i_2} \dots$ not involving the first j letters. The required simplicial complex $K(B)$ is obtained by associating to each such monomial the simplex $\{i_1 - j, i_2 - (j-1), \dots\}$. *q.e.d.*

The shifted simplicial complex $K(\Delta_{\mathsf{F}}(B(K)))$ is called the *symmetric shift* of K. The symmetric shifting operation

$$K \rightsquigarrow \Delta_{\mathsf{F},s}(K)$$

shares many of the properties of exterior shifting, e.g.

(c) *Any simplicial complex K has additive (co)homology over F isomorphic to that of its symmetric shift. Further, K is Cohen-Macaulay if and only if its symmetric shift is pure.*

Proof. Both $B(K)$ and $B(\Delta_{\mathsf{F},s}(K))$ have the same symmetric shift. So this follows from Theorems 1, 2, and part (b). *q.e.d.*

The second part of above result is a reformulation of Reisner's theorem, whose original proof involved Kozsul resolutions of rings. The purity of the shifted complex being the key element in the proof of the Upper Bound Theorem of §9, it can thus also be (as indeed it was first) deduced from Reisner's theorem.

(2) However note that *the symmetric shifted simplicial complex can be different from the exterior shifted simplicial complex.*

For example, for the Kuratowski 3, 3 graph this is so. To see this note that

...

We now go on to finer results (but still analogous to those of §8 for exterior shifting) using an equivariant version of symmetric shifting. Formerly, only some of these were known, and that too as consequences of results concerning (co)homology of toric varieties, a topic treated in §15 below.

comment

As against this, the argument for the sharp Heawood inequality will exploit the fact that the $\mathbb{Z}_2$-action of $E = K_*$ is actually free on all simplices.

To do this consider the quotient $\overline{\mathcal{E}}$ of $\mathcal{E}$ (or even of $(\mathcal{E})_o$) obtained by declaring two semi-simplices of $\mathcal{E}$, of the same type, as having the same (*resp. opposite*) *typed-orientation* if each can be obtained from the other by an even (*resp.* odd) number of transpositions of the letters used to denote the vertices.

THEOREM 5. PROPERTIES OF $E \rightsquigarrow \overline{\mathcal{E}}$.

(a) *A van Kampen obstruction class* $\mathfrak{o}$ *of a free* $\mathbb{Z}_2$*-simplicial complex* E *vanishes if and only if the corresponding characteristic class of* $\overline{\mathcal{E}}$ *also vanishes.*

Proof. This follows because any total ordering of the pairs of antipodal vertices of $\mathcal{E}$ determines a $\mathbb{Z}_2$-section of the quotient map $\mathcal{E} \to \overline{\mathcal{E}})$, *viz.* that which maps any semi-simplex to the sequence of its vertices determined by the induced total ordering. *q.e.d.*

Thus, despite being much smaller, $\overline{\mathcal{E}}$ does not lose the information in which we are interested. Besides, it can be characterized algebraically in the following very pleasant manner.

§7. Symmetric shifting.

Combinatorial shifting, though quite explicit, suffers from the disadvantage that it is a step-by-step procedure, and the inductive checking of hypotheses is not easy. The shifting operations of this, and the next section, are one-stroke operations, and were discovered by Kalai.

For example recall that in §6 we considered the complex L of semi-simplicial chains of K. [*Important.*We will now only consider field coefficients F, and that too of the type (say $\mathbf{R}$ or $\mathsf{F}_2(X_1, \ldots, X_{N^2})$) which have *transcendence degree at least* $N^2, N = |vert(K)|$, over their prime subfield.] There was the associated *canonical* vector space basis given by the semi-simplicial complex $B(K)$, i.e. the order ideal of all monomials, in the vertices x_i, supported on simplices of K. We will show how this can be replaced by another shifted semi-simplicial complex $\Delta(B(K))$ in one go.

More generally this operation

$$B \rightsquigarrow \Delta(B)$$

will take semi-simplicial sets to semi-simplicial sets and is defined as follows.

Consider the vector space $L(B)$ of all finite linear combinations of the given set B of monomials $\prod x_i^{q_i}$ in x_i, $1 \leq i \leq N$. Now switch to $\theta_i = \sum \alpha_{ij} x_j$, where the α_{ij} are *algebraically independent* over the prime subfield, and select, from the *generic* vector space spanning set of *all* monomials $\prod \theta_i^{q_i}$, the vector space basis $\Delta(B)$ which is least in the lexicographic total ordering of these monomials.

THEOREM 1. *The operation* $B \rightsquigarrow \Delta(B)$ *maps semi-simplicial sets to shifted semi-simplicial sets. Further, it preserves their face vectors,* *commutes with inclusions, and shrinks shadows. So if B is an order ideal of monomials, then so is $\Delta(B)$.*

Proof. ... ... **q.e.d.**

For the above case $B = B(K)$ the face vector of B is of course the h-vector of K. Further as we saw in §6 there is a one-one correspondence $B \rightsquigarrow K(B)$ from shifted semi-simplicial sets to shifted simplicial sets, with inverse $K \rightsquigarrow B(K)$. ???

Just as for the combinatorial case we will now equip this operation with suitable maps and then consider its homological properties.

§8. Skew symmetric shifting.

Instead of constructing from K a shifted model by

$$K \rightsquigarrow B(K) \rightsquigarrow \Delta(B(K)) \rightsquigarrow K(\Delta(B(K))$$

as in last section, we can do it alternatively as follows.

Define the operation

$$K \rightsquigarrow \Delta^{[1]}(K)$$

simply by noting that $L(K) = L(B(K)$ has the summand $C(K) = L^{[1]}(K)$. We cut down on the generic spanning set by only keeping those in which each θ_i occurs with degree one, and *then* choosing the first lexicographic basis.

Instead of [1], the same game can be played with any [r], and results in $\Delta[r](K)$ corresponding to the shifted semi-simplicial set of monomials occuring as the smallest generic basis from the smaller spanning set of monomials in which no vertex occurs with degree $\leq r$.

To see the behaviour of these operations with respect to total homology one needs to look at some chain maps.

endcomment

(1) Coboundary defined by star gives same homology as that of the singular linear simplex. So preservation of this equivariant cohomology suffices to look at the vKO's. ok.

(2) Purity for the star-shift of a simplicial sphere will follow because a lesser dimensional top simplex's star will be an invariant cycle not bounding an invariant chain. ok.

(3)

Note

15. 4. 92.

(1) Enclosed is an extract* from chapter 5 of "Van Kampen Obstructions" (a research monograph *still* very much under preparation ... date of completion *still* very uncertain ...). This chapter is entitled 'Heawood Inequalities' and explores at length (the interrelationships between) the various methods used to prove inequalities like the original one of Heawood (or is it Kempe?), viz. that a graph embeds in the plane only if the number of its edges is less than three times the number of its vertices.

(2) More immediately the enclosed section '8 bis' (still not written out in full) will constitute the middle and main section of a revised version of my paper, "Shifting and Embeddability", which should be ready pretty soon, *in case* the enclosed arguments are correct ....

(a) The rest of '8 bis' is pretty easy and being written down : I have an equivariant cochain isomorphism from the equivariant cochains of the original, to those of the new star shifted complex .... throwing the van Kampen obstruction of first, to that of the latter ... and also should be able to check easily that if the original complex is closed under upper star, so is the new one ... leading to at least the implication that vanishing of the obstruction implies absence of 'Kuratowski complexes' in the types under question .... (... the rest of the method, converse implication involving classification etc., I will develop in a sequel to this paper).

(b) The first short introductory section of my new "S and E" will be in the same spirit as before, except of course that I will now be able to state a much better theorem (viz. direct part of a higher dimensional Kuratowski theorem) and so better immediate corollaries: e.g. the absence of a particular subcomplex yields at once the sharp Heawood inequalities etc. ...

(c) Also, as before, in the concluding section of "S and E", I will outline things being explored further ... (connections to Sullivan's work, Whitten's deformation of de Rham derivative, cyclic cohomology, etc.) ... and also mention, giving a summary account as in the enclosed section '9', Kalai's argument – published already in his "The Diameters of graphs of convex polytopes and f-vector theory," and depending also on Kalai's theorem, from his "Algebraic shifting methods ..." (??), that exterior shift of a sphere must be dimensionally pure – by which the absence of the aforementioned subcomplex alone yields also, for the case of simplicial spheres, the still sharper inequalities conjectured by McMullen ....

*pp. 38-56

In case the enclosed (which is the key part of the whole thing !)is correct, this new version of "S and E" should be ready pretty soon ...

So I'll appreciate if you let me know any shortcomings/mistakes etc. in the enclosed extract ...

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