(part three and a half : translation of ਭਾਗ ਸਾਢੇ ਤਿਣ)

50. Clearly in the following, from my notebooks, there are items which could have easily been typed into the main text. But most are now half-forgotten to me, and may need quite some work. Neither are they in any particular order, and quite a few have as such no relationship with mathematics. Nor will I be polishing these here, but maybe somewhere else ...

(50.01) Left and right switch in a mirror, why not top and bottom?

(50.02) A reflection of a line in a point, we can extend by half-turn of the plane around that point. This preserves the orientation of the plane, while the line's orientation was reversed by the reflection. Khayyam's method had used reflections, in the method of part four we'll use half-turns.

(50.03) Real polynomials with all roots real (union of all n-swallowtails)'s recognition and *unimodality,* circle action so *cyclic homology,* maybe also a *cyclic shifting,* starting Neighbourly Triangulations (1983) some *easier riemann hypotheses,* see especially Simplicial (Co)homologies (1994), but note that the from it begotten paper of 1998 has alas no cyclic homology, old work of Jensen and Polya showing Riemann's hypothesis is equivalent to many similar easier hypotheses, a 2019 method of Zagier confirming many easier ones, the efforts of Connes to obtain Riemann Hypothesis using cyclic homology, etc.

(50.04) Complexification at once makes the n-swallowtail's topology hard, like now even the affine's fundamental group is something to reckon with, it surjects on the group of $n!$ permutations, 'birthing' galois theory. Far simpler is the real n-swallowtail that we can 'see', and projectively even it has S^1, in fact baby action. It would be far more natural if using cyclic homological methods now we could find a way till where presently only complexified baby–meaning that of Picard, Lefschetz and Kähler–action has taken us. For example, there is imminent hope that at any time my Heawood Inequality Conjecture and many similar problems will be resolved, but in my opinion for *a clear understanding* we should use only real swallowtails and baby action.

(50.05) It is very wrong to dub what is in Euclid geometry in current usage of the word. Yes, till a hundred or so years geometer and mathematician were synonyms, but now geometry has been reduced to a part of mathematics, and one even meets some saying maths needs no geometric intuition! To view mathematics so formally is also Euclid's gift. And, the results of Eudoxus in the fifth book were for Dedekind the definition of $\mathbb{R}$ with which rigorous analysis books of today start. And, the school proof from Euclid that $\sqrt{2}$ is irrational, did not need much more as Gauss saw to show that if in fact any algebraic integer is rational then it must be an ordinary integer, etc.

(50.06) Not only rulers, the definitions of adding and multiplying rulers, calculus too, are all observer dependent things, means with which these local beings are seeking a global truth (perhaps involving cross-ratios and relativistic distance of a closed spherical submanifold or of a bounded open set) on or in which they are living different lives. But we beings have become so used to some one and two dimensional rulers–real and complex numbers–that often to

us the relation between this geometry and these from it born numbers seems to be like that between an egg and a hen.

(50.07) Ahlfors is best for complex cross-ratios and most anything complex, but as such has not hyperelliptic integrals. That are in for me even dearer book of Goursat. Which had started me on this journey, see P G & R, note 5 (by the way the seven notes on that single page still seem most important to me, the rest is only a continuing attempt to clarify them). Very patiently Goursat explains that truly it is really all about understanding the simple integral $\int \frac{dz}{\sqrt{1-z^2}}$ well – see second volume, page 50 to page 60 – once we have done that, we grasp at once what means any hyperelliptic period.

(50.08) In this journey there has been for sure one unity, it is of one, mine. Only myself I'm trying to understand and (can and) am trying to teach: the reason why there are so many links to my own papers! Because of the last ten years' work, my once totally disparate old papers have also come under that one roof, the roof for which I have been using adjective cartesian : after it you can put either mathematics or physics, both are the same for me! Certainly I must be invoking the name of Des Cartes wrongly. I'm no scholar of his philosophy. Yes, many decades ago for a while I had read him a lot. It seemed to me he had set us off on the right track. Later we forgot this simple way. My journey is in search of forgotten simplicity.

(50.09) Which garden radish am I? Even for someone like Descartes it became necessary to seek solitude in which he could understand his journey correctly. For about a dozen years whatever maths Keerti, Jitender, Dinesh, Vibhor et al have now and then sent me, I've limited my answer usually to just a warm thank you, keep it up, congratulations, etc. Despite the fact that I did read almost all these papers, and often ideas too came to me which could have been useful for them. But I know of my habit, that if I start the talk then at once I'll start myself, dragging them along, down another and for me not necessary path. Leave alone parallel-processing and multi-tasking, often it becomes hard for me 'to walk and chew gum at the same time'!

(50.10) 'Re-inventing the wheel' is not a bad thing : there is much to be said for a fresh eye! For example, our affine complex n-swallowtails, spaces of all degree n equations with n distinct roots, are also configuration spaces of n alike but always separate moving particles in the plane, which tells us further that its fundamental groupoid, that is the braids of Artin, are directly tied to our theory of equations. His braid groups give the full homotopy type of these swallowtails, and a result of Arnol'd their full cohomology.

(50.11) Burnside & Panton's old book, mod numerical methods, minus most formulas, plus enough topology, is our theory of equations: without this building the obstruction theory of Galois is like a cheshire cat's laughter!

(50.12) Even the part of the building tied to quadratic equations from school is very beautiful–Möbius strip, $\mathbb{R}P^2$, $\mathbb{C}P^2$, $T^1(S^2) \cong \mathbb{R}P^3$ etcetra–and reminds me of some 'homework' sent to me in Chandigarh from far away Princeton in 1969 by Professor Steenrod : my story would have been quite different without

his kindness. And also comes to my mind a beautiful paper on symmetric powers with some others by Živaljević : if there is one another who can understand all my mathematics right it perhaps is Rade.

(50.13) Mathematics done far in time is as important to comprehend as that being done by researchers sitting far away. Mathematicians my age often read contemporary, and usually formally written, work in French (and German and many in Russian too), but there are very few who have ever read the book in which Galois's ideas were worked out fully, Jordan's "Traité" of 1870. The mathematics done a hundred and fifty years ago (even if it be in the mother tongue of the reader) is not easy to understand, but there is a lot here, like that theorem on its page 380, which is nowhere else.

(50.14) I have not written the nationality of the workers. The violence of the past, and that still going on today, has been due even more to nationalism than religious fanaticism. For how unnatural, unreasonable and ugly is this partition of the earth into countries read "The Iks" by Lewis Thomas an essay from this biologist's book "The Lives of a Cell" (1974).

(50.15) The adjective "great" was also reduntant. For first, all mentioned have done very good mathematics. Second, I've now an allergy to this word because of its common misuse, see Sohn Lal s/o Vaso Mal.

(50.16) "But does that not reduce your readership?" one Punjabi asked me when I told him that these days I was writing my mathematical thoughts only in Punjabi. If you are with me so far clearly you are a Punjabi speaking and reading mathematician like me. So for sure you can read mathematics written in two or three Western languages, and most probably write it in one. But the nub of the matter is, when you *do* mathematics—or now and then talk about it with another like you—you *think* in Punjabi only. Yes of course, besides technical terms you likely use foreign phrases and words in common usage too. This only, the natural language for all working Punjabi mathematician like me, is what I'm writing these essays in. Besides I've been trying, for those not knowing Punjabi, to post an English translation in three months. In fact, there has been an increase in the number of those fond of regularly reading my mathematics, ever since I adopted Punjabi policy!

(50.17) If a language does not succumb to the whims of fashion or of those in power, it becomes more powerful than the dominant language! All educated Punjabis, let us say all those who can read and understand Punjabi Tribune fully–see Switches and Fingers (ੜ)–they must surely know English too. Limiting a language to just stories and poetry – as "Urdu lovers" of the India of today want it to be limited – is as good as saying goodbye to that language! I had no difficulty at all, and neither should any other Punjabi mathematician have, in writing mathematics in this powerful language.

(50.18) The compositions of Gurus and Bhagats in the SGGS are in Punjabi and more than a dozen other languages, but all in Gurmukhi script of those times (like there are very few 'bindis' in the Gurbani). Punjabi language too is written in Gurmukhi and other scripts. For example, the Heer of its shakespeare

Waris Shah is in Shahmukhi, a script which alas I and almost all living in eastern Punjab don't know. On the other side, Punjabi language is mostly written in Shahmukhi, and people don't know Gurmukhi script.

(50.19) All our scripts are fairly phonetic. If a human or program has been taught two, there should be no difficulty in transcribing one into the other. For here the linguistic meaning of words or technical terms used has no rôle at all, only the same sounds are to be shown in the new script. In fact, useless data about language or spell-checks, dictionaries, etcetra will not only make such a light and swift app heavy and slow but also bad.

(50.20) Learning Shahmukhi has been for long in that in-tray of mine, and neither have I learnt of any good transliteration app in this time. Yes, there is a lumbering program Aakhar having this facilty, but when I asked it to change some words from Gurmukhi to Shahmukhi and back to Gurmukhi they came back mangled : ਟੋਪੋਲੋਜੀ → ٹو پولو جی, ਸਵੈਲੋਟੇਲ → سو ولو ٹیل, ਪਵਾਂਕਾਰੇ → پوا نکارے ! Thankfully however ਥੀਓਰਮ → تھیورم maybe because theorem is spelled thus in some ਸ਼ਬਦਕੋਸ਼ loaded on it? In the just linked paper is my experience of some years ago with our dictionaries, and in the previous linked paper is described what care we are taking of some records more precious than the Koh-i-noor that have so far survived largely by chance only.

(50.21) Translation from one language to another – even if both are in the same script, and the translator expert in both languages – can be difficult. For example, can someone *please* give me a literal translation of the records in Farsi Siyakat and a Shikasta hand from the treasury department of the Sarkar Khalsa attached to the paper above about Koh-i-noors?

(50.22) Recalling that in the wake of the Sputnik the AMS had published some quite amusing translations of mathematical papers from Russian, I'm not about to leave translation of my mathematical papers in Punjabi to the auspices of this kind society! Boring the job of translation is, but it is better in my opinion that this work too a researcher should do himself.

(50.23) I'm very grateful to Library Genesis. I can't understand at all why these kind people who are providing this service daily to thousands of researchers on the internet have not yet been given a Nobel Peace Prize?

(50.24) There are many other odds and ends in the paper to which I linked in (50.17) above! Like in its (ठ) there is a recollection of my old and dear higher Heawood inequality conjecture. In December 2018 on Kalai's blog has appeared the sensational news that it has now been proved by Adiprasito! The details I've not yet read, but clearly it is by using many means and power that Karim has solved this hard problem: congratulations again young man!

(50.25) On July 22, 2019 this website will complete ten years, and it seems more than twelve thousand beings or bots have visited it so far! Despite or maybe because with God's blessings its still "under construction".

(50.26) We have "historians" galore forever shooting the breeze, but from this scrutiny into an event of 1915 and the Koh-i-noors of (50.20-21) it is clear: the Punjab badly needs a young, intelligent and diligent historian!

(50.27) But I note my Poincaré Seminar and papers from it is essentially all mathematics–not its history–written for working mathematicians.

(50.28) Besides work forgetting set theoretic jargon and the myth of modern rigour is needed if we want to learn something new from an elder. For example, in the language of abstract groups left and right cosets have equal status, so I started doubting the proof of the theorem on page 257 of the "Traité" of Jordan. A full two days later it dawned this equality springs from an unnatural language, the natural word is orbits of a subgroup, left and right depend solely on whether we write the same composition of permutations fg or gf.

(50.29) Arnol'd's "Catastrophe Theory" has hand-in-hand with excellent new mathematics his thoughts on this jargon, myth, Poincaré, learning-teaching, etc. Very fortunately for fourteen years now I'm only taking my own class–rare are the opportunities now for dusting other ankles off!–but you will find in full my own philosophy of learning-teaching in the easiest two pages of the following ten-year old paper, read its Intermission.

(50.30) "213,16A" and Mathematics is among my best papers. The 'Notes' mentioned in it eventually I had not written, and neither are they necessary to follow it. But, despite its appearance, it is a fairly deep mathematical paper, and as any mathematician knows, quite often to understand just one page of mathematics it can take us many days.

(50.31) Mathematics is big but one! The above paper had some number theory too, and in ਬੀਜ ਗਣਿਤ ਦਾ ਮੂਲ ਮਸਲਾ we tied its Möbius's balcony to school algebra; moving on in this story we'll see its Four half turns tilings solve all degree four equations by elliptic functions; and somewhat similarly those tied to Magic carpet all fifth degree equations; etcetra.

(50.32) Towards the end of this same 2010 paper I and let us say my alter ego – besides there are also two sweet little sisters and an even littler but far stranger bing in this totally true short story – begin doubting the very existence of any Möbius strip, and in this process remember again a basic truth about all mathematics. It was due to bing only that when Aravinda asked me in 2019 if he could print this paper in his magazine Bhāvanā I said yes but that the title The Elves of Mathematics would be better.

(50.33) In Extracts from 2008 there is another amusing story of a Hero from Alexandria who went to Rome and used the poetry of Virgil to forever tie Queen Dido of Carthage to mathematics; and how this story ties to Four half turns and a wrong method for finding area is also in this paper.

(50.34) With admirable modesty Jordan in the preface of his "Traité" of 1870 calls this great book but a commentary on the ideas of Galois, after telling us Galois had mentioned another new attractive application of his groups. It is only with the third Livre or page 243 that Galois theory starts, with the definition of his groups of an equation, algebraic as well as monodromic, see pages 258 and 277. And in quick order Jordan takes us on a full tour of this new attractive way, on which already had set out Hermite, Kronecker, Clebsch and many more. Showing striking strength – see especially from page 348 to

the end of this Livre – this youth, after reproving one after another the results of all these seniors with groups, suddenly on page 380 gives, *a Galois theoretic method for solving any equation using hyperelliptic periods!*

(50.35) Alas! almost nothing above is to be found in any Galois theory text book of today. But yes, believe me, that which is in most such books, that too is in this 1870 book, only not written in set-theoretic jargon.

(50.36) Often I have talked of the vector space $\mathbb{R}^\infty$ of all sequences with almost all entries zero, but not engaged so far with the various sequence spaces of Lebesgue, Banach et al that arise from the *gauge* of its various balls B^∞. The most natural and biggest is the space of all real sequences with the topology of termwise convergence F of Fréchet. $\mathbb{R}^\infty$ *is dense in* F :- any sequence $s \in F$ is the limit of its truncations $s_n \in \mathbb{R}^n$. □ *There are translation invariant distances defining the topology of* F :- e.g., $d(s,t) = \sum 2^{-n} \frac{|s_n - t_n|}{1+|s_n-t_n|}$. □ Even though F is an LCTVS because its topology is defined by the seminorms $s \mapsto |s_n|$, *any ball* $B^\infty = \{s : d(0,s) < 1\}$ *is not convex,* because then its gauge would be a norm, *but* F *has no continuous norm* :- for then there are $u_n \in F$ with norm 1 and only nth entry nonzero, but the limit of u_n in F is zero. □

(50.37) *Three simple principles*—intermediate value theorem, correct local fitting of the tiles, and S^2, $\mathbb{R}^2$ or B^2 being simply connected—had enabled us in that 2010 paper to raise armies of regular tilings (so groups)! Now we only require the union of any two tiles sharing an edge be symmetric around its midpoint, *but now, the given* $p > 3$ *points on the initial circle are arbitrary.* Using the same three principles shall now give us for each multiple q of p, such a tiling $\{p, q\}$ moving continuously with the initial data. Further save $\{3,3\}$, $\{3,6\}$ and $\{4,4\}$ they are all relativistic, that is, are tilings of a moving disk B^2 of a finite radius $c < \infty$ depending on the initial data.

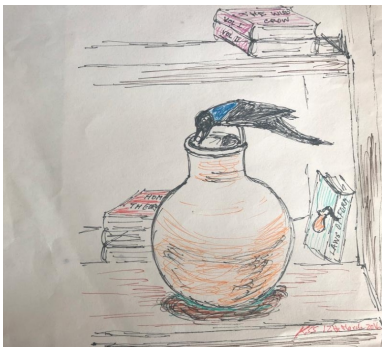
(50.38) When the seed polygon of an $\{n, n\}$ tiling has *a diameter as one side,* it has an antipodal adjacent tiling and their union—a $2(n-1)$-gonal fundamental region of the index two *period subgroup*—is also inscribed in S^1. These periods identify pairs of opposide sides of all unions to make a torus for $n = 4$, a surface of genus two for $n = 5, 6$, of genus three for $n = 7, 8$, etcetra.

(50.39) The geometric theory of equations perhaps starts from degree three only with ∞ admitted as a possible root. Two additional and lowest roots from our pocket called $0, 1$ can serve to Cayley measure the unknown roots of any given equation of the real n-swallowtail with a third ∞ very far indeed to rotate cyclically all roots. Móbius geometry is now complete, meaning those equations from schools with reciprocal roots are also in play and shall give us up-down *Poincaré duality.* When using rigidity we view these special equations within the Möbius geometry of the plane, the $(3 + n)$-gon of roots is *special* in the sense of (50.38), the horizontal diameter being a side. And the *first Betti number* of the surface that gets made by dividing by the index two subgroup of all even compositions of half-turns of the tiling is the dimensional ambiguity of an integral made from the equation. Beyond homology we have too the vector basis formed by its *Cauchy periods* : from these we can compute the hyperbolic

translations determined by the doubled $2(2+n)$-gon in the unit circle, so upper $(3+n)$-gon also, so the unknown n roots.

(50.40) In this context comes to my mind Atiyah saying that the celebrated Index Theory of his with some others was in fact only Cauchy's integral formula! His last papers are replete with many more such pithy pointers that can be very helpful, so in my opinion these fairly wrong papers of this great mathematician are worth preserving and reading.

(50.41) Out in Kansas my second hero Grothendieck–see portrait–wrote that crystal clear Sheaf Theory from which and its consequent Tohuku I had gained much assurance: *this way even I can do mathematics!* But later on to understand him fully it became necessary to understand also the disciples attending his SGA near Paris which was beyond me sitting alone far away.



(50.42) Many years later I too was visiting Bures-sur-Yvette often. It seemed even the leaves in the jardin surrounding the institut were imbued with the scent of its mathematics! But alas the demon of nationalism didn't spare that country either, restrictions were imposed even on the dress of some citizens. These visits now no longer remained means of relief for me, so like my hero I too bade adieu to this temple of mathematics.

(50.43) Because of this demon's ever-increasing popularity and conversely : songs like the Universal Soldier of Buffy Sainte-Marie are rare!

(50.44) From the numbers and graphs of today 11/04/20 it does not seem a miracle will occur here–only the onset of the virus was later in South Asia, its rate of spread is the same–and there is danger of the same number of deaths as in the holocaust of 1947, possibly more than the Spanish Flu of 1918. Despite demon my prayer to the Almighty is that the survivors will teach their children not nation first or religion first but : truth first!

(50.45) The last link and the next three notes were added later, here were these memos from my notebook: analysis of the orbit space of the S^1-action and Connes homology. The fundamental group of the complex swallowtail, Artin's braids. Along with symmetric polynomials also skew-symmetric forms à la Weil and Chern. Forsaking complex to search cyclic real methods for higher Heawood inequality etcetera is more natural than like Selberg forsaking complex to find fully combinatorial methods.

(50.46) But as often happens we make plans but what is acceptable to God is something else : odds and ends left after Part Five kept me busy till this end of 2020, and only some days ago have I posted this much from these : quite a few more things are more clear now. But clearly this mathematical panorama, that my papers have opened, is not going to be tamed by just one cartesian, but for the time being this one-man army is at it!

(50.47) There are individual items of great interest here, like how to any cyclic sequence of nonzero nonequal complex numbers are associated Poincaré's Kleinian functions – now called automorphic functions, they are still easiest to learn directly from his masterly work of 1881 – amongst them those inverting hyperelliptic integrals whose periods we can find from the coefficients only of the equation of which the numbers of the sequence are roots.

(50.48) But there is too a unity in this theory till its roots, not only are the above functions a response to the fundamental problem of algebra (2017), in making them only comes to the fore the need of a relativistic geometry on any open bounded set Ω of the plane, see PG&R (2013), Note 1.

(50.49) Some years ago reading my mother's gutka I noted that in Japji Sahib one and only one bindi is used—in the word ਭਾਂਡਾ of the last pauri—although there are many other words in it for which we now use bindi.

ਭਾਂਡਾ ਭਾਉ ਅੰਮ੍ਰਿਤੁ ਤਿਤੁ ਢਾਲਿ॥
ਘੜੀਐ ਸਬਦੁ ਸਚੀ ਟਕਸਾਲ॥

(50.50) Whatever I have been able to achieve I owe above all to the love and affection of my mother Bhupinder Kaur. She was born a hundred years ago in 1919, perhaps on February 17, and passed away on May 18, 1995. This by God's grace continuing work is dedicated to her memory.

K S Sarkaria from September 30, 2018