ADDENDUM TO MY PAPER "ON COLORING MANIFOLDS"

BY

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An important paper by Grünbaum [1], which had escaped my attention until now, contains the following theorem: If \( m \geq 2 \) then one can assign \( 6(m - 1) \) colors to the \((m - 2)\)-simplices of any simplicial complex imbedded in \( \mathbb{R}^m \) in such a way that any two \((m - 2)\)-simplices incident to the same \((m - 1)\)-simplex have different colors. A fortiori, this implies the finiteness of the numbers \( \chi_{m-3}(S^m) \) of [2].

It is easily seen that Theorems 1 and 2 of [2] are equivalent to the following.

**Theorem A.** If \( X \) is any closed \( m \)-dimensional pseudomanifold \((m \geq 2)\), then

\[
\chi_{m-3}(X) \leq \left\{ \frac{m(m + 1)}{m - 1} [1 + b_{m-4}(X; \mathbb{Z}_2)] \right\}.
\]

Further if \( K \) is any subcomplex of a triangulation of \( X \) and contains at least one \((m - 2)\)-simplex, then

\[
\frac{m - 1}{m + 1} \alpha_{m-3}(K) \leq \alpha_{m-2}(K) + b_{m-4}(X; \mathbb{Z}_2) - 1.
\]

We will now use the ideas of Grünbaum [1] to show that this theorem can be significantly improved when the hypotheses are strengthened somewhat.

**Theorem B.** If \( X \) is any closed triangulable manifold \((m \geq 3)\), then \( \chi_{m-3}(X) \leq 6 \). Further if \( K \) is any subcomplex of a triangulation of \( X \) and contains at least one \((m - 2)\)-simplex, then \( m \alpha_{m-3}(K) < 6 \alpha_{m-3}(K) \).

**Proof.** The first part will follow from the second (as in the proof of Theorem 2 of [2], for example). Let \( K \) be a subcomplex of a triangulation \( L \) of \( X \) and let \( \sigma_1, \sigma_2, \ldots, \sigma_t \) be the \((m - 3)\)-simplices of \( K \) which are incident to at least one \((m - 2)\)-simplex of \( K \). Since \( X \) is an \( m \)-manifold \((m \geq 3)\), \( \text{Lk}_i \sigma_i \), \( 1 \leq i \leq t \), is a triangulation of the 2-sphere \( S^2 \). Further \( \text{Lk}_i \sigma_i \), \( 1 \leq i \leq t \), is a subcomplex of \( \text{Lk}_i \sigma \) and contains at least one vertex.

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Applying the case \( m = 2 \) of Theorem A (or Lemma 1 of [1]) one gets
\[
\alpha_i(L_k\tau_i) \leq 3 \alpha_0(L_k\tau_i) - 3, \quad 1 \leq i \leq t.
\]
Adding these inequalities one has
\[
\left( \frac{m}{m - 2} \right) \alpha_{m-1}(K) \leq (m - 1) \alpha_{m-2}(K) - 3t
\]
which implies \( m \alpha_{m-1}(K) \leq 6 \alpha_{m-2}(K) \).

Thus the "finiteness theorem" stated in the introduction of [2] can be improved to the above "six color theorem"; however the above proof does not generalize to pseudomanifolds \( X \).

For any compact triangulable space \( X \) let us denote by \( Ch_i(X) \) the least number of colors which suffice to label the \( i \)-simplices of any triangulation of \( X \) in such a way that distinct faces of an \((i + 1)\)-simplex are assigned distinct labels. It is clear that \( Ch(X) = Ch_0(X) \). We can use Grünbaum's trick of using "weight functions" (see [1]) to supplement Theorem B with the further assertion that for any closed manifold \( X \) of dimension \( m \geq 3 \), \( Ch_{m-3}(X) \leq 6(m - 1) \). The same trick and Theorem A can be used to get upper bounds for \( Ch_{m-3}(X) \) when \( X \) is an \( m \)-dimensional pseudomanifold.

Further results and conjectures. We have proved that if \( X \) is a compact triangulable space with dimension greater than or equal to \( 2t + 3 \), then \( Ch(X) = \infty \). Another result of some interest is that \( Ch_{m-1}(X) = 2 \) whenever \( X \) is a closed manifold with dimension \( m \geq 2 \). We hope to give elsewhere a proof of the fact that \( Ch_i(X) \) is finite whenever \( X \) is a closed manifold with dimension less than or equal to \( 2i + 2 \). In view of Theorem B above it seems likely that the number \( Ch_{m-3}(X) \) is the same for all closed \( m \)-dimensional manifolds \( X \) with \( m \geq 3 \); quite possibly the numbers \( Ch_{m-2}(M^m) \) are the only ones which reflect the global topology of a closed manifold.

If \( X \) is a closed triangulable \( m \)-manifold \( (m \geq 3) \), then \( Ch_{m-3}(X) \leq 4 \); this improvement of the first part of Theorem B can be obtained by using the four color theorem.

Added in proof. For more discussion regarding results mentioned above see [3] and [4].

References

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