SHORTER NOTES

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EMBEDDING AND UNKNOTTING OF SOME POLYHEDRA

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ABSTRACT. If a compact polyhedron $X^n$, $n \geq 3$ (resp. $n \geq 2$), has the property that any two of its nonsingular points can be joined by an arc containing at most one singular point, then $X^n$ embeds in $\mathbb{R}^{2n}$ (resp. unknotted in $\mathbb{R}^{2n+1}$).

The object of this note is to discuss a situation where the Penrose-Whitehead-Zeeman construction (see Zeeman [10, pp. 66–67]) works for a class of polyhedra much more general than manifolds. In particular reduced polyhedra satisfy our hypotheses. Thus Husch's unknotting theorem [4] is a special case of the result proved below.

Let $X$ be a compact polyhedron of dimension $n$. A point $x$ of $X$ is called nonsingular (resp. singular) if there exists (resp. does not exist) a triangulation of $X$ containing $x$ in the interior of an $n$-simplex.

THEOREM. Let $X$ be a compact polyhedron of dimension $n \geq 3$ (resp. $n \geq 2$). If any two nonsingular points of $X$ can be joined by an arc containing at most one singular point, then $X$ embeds in $\mathbb{R}^{2n}$ (resp. unknotted in $\mathbb{R}^{2n+1}$).

Embedding. General position yields a p.l. map $f: X^n \to \mathbb{R}^{2n}$ with a finite number of nonsingular double points. To explain our iterative construction it suffices to consider the case when there is just one pair $\{x_1, x_2\}$ of nonsingular double points, $f(x_1) = f(x_2)$. Let $A$ be an arc, containing at most one singular point of $X$, and joining $x_1$ to $x_2$. Because $2 + n < 2n$ any general position point $p$ of $\mathbb{R}^{2n}$ is joinable to the circle $C = f(A)$ in such a way that the $2$-disk $D = pC$ meets $f(X^n)$ in precisely $C$. By choosing triangulations of $X^n$ (resp. $\mathbb{R}^{2n}$) in which $A$ (resp. $f(X^n)$ and $D$) are full subcomplexes, and $f$ is simplicial, we can find regular neighborhoods $N(A)$ of $A$ in $X$, and $N(D)$ of $D$ in $\mathbb{R}^{2n}$, such that $f(X - N(A)) \subseteq \mathbb{R}^{2n} - N(D)$, $f(\partial N(A)) \subseteq \partial N(D)$ and $f(N(A)) \subseteq N(D)$. If $A$ has no singular point, $N(A)$ is an $n$-disk. If $A$ has the unique singular point $y$, then $N(A)$ is p.l. homeomorphic to the closed star of $y$. In either case we see that $N(A)$ is a cone over its boundary $\partial N(A)$. Therefore we can extend the embedding

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f | (X^n - int N(A)) to an embedding g: X^n → R^{2n} by coning f(∂N(A)) over an interior point of the 2n-disk N(D).

Unknotting. General position yields a p.l. map f: X^n × I → R^{2n+1} × I whose 'ends' f_0, f_1 are two given embeddings of X^n in R^{2n+1}, and which has a finite number of nonsingular double points. Since any two nonsingular points of X^n × I too can be joined by an arc having at most one singular point, we repeat the above construction to get a p.l. embedding g: X^n × I → R^{2n+1} × I with ends g_0 = f_0, g_1 = f_1. Thus f_0 and f_1 are concordant. By Lickorish [5, Theorem 6], concordance implies isotopy in codimensions ≥ 3. Thus f_0 and f_1 are isotopic.

The above theorem is best possible in the sense that one cannot replace 'at most one' by 'at most two'. Recall that two n-spheres can link in R^{2n+1}. Thus, by joining two n-spheres, n ≥ 1, by a thin 'ribbon', we get an example of an n-dimensional polyhedron which knots in R^{2n+1} and for which any two nonsingular points can be joined by an arc containing at most two singular points. Another example of a polyhedron having this joinability property is the n-skeleton of an N-simplex, N ≥ 2n + 1, n ≥ 1. It was proved by van Kampen [7] and Flores [3] that, for N ≥ 2n + 2, this polyhedron does not embed in R^{2n}.

Husch unknotting. A homogenously n-dimensional and connected polyhedron X^n is called reduced if it can be obtained from some other, Y^n, by replacing a regular neighborhood N(T), of a maximal tree T of a triangulation of Y^n, by a cone z · ∂N(T). Since T is a maximal tree, for each x ∈ Y we can find a t ∈ T and an arc α from x to t such that all points of α − {x, t} are nonsingular points of Y^n. From this it follows that any nonsingular point of X^n can be joined to ∂N(T) via nonsingular points of X^n, and thus, that any two nonsingular points of X can be joined by an arc A through the base point z, such that all points of A − {z} are nonsingular points of X^n. Therefore the above theorem implies Husch's result [4] that all reduced polyhedra X^n, n ≥ 2, unknot in R^{2n+1}. Note that the n-skeleton of a 2n-simplex, n ≥ 2, is not reduced, but does satisfy the hypothesis of the above theorem.

Bibliographical remarks. The case X^n = a connected pseudomanifold (resp. X^n = polyhedron obtained by making some identifications on the boundary of a connected manifold) of the above theorem is due to van Kampen [7] (resp. Edwards [2]). The construction given in the above proof (resp. general Penrose-Whitehead-Zeeman construction) is a variation (resp. a generalization) of a construction by which van Kampen [7] eliminates those pairs of double points, of a g.p. map f: |K^n| → R^{2n}, which lie in adjacent n-simplices of K^n. For other developments of van Kampen's ideas see also Shapiro [6], Wu [9] and Weber [8]. For more on singularities see Akin [1].

REFERENCES


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