1. Introduction

(1.1) It is well known that if a 1-dimensional simplicial complex, i.e., a "graph", $K^1$, embeds in a 2-dimensional manifold $M^2$, then its chromatic number is less than a certain constant $c$, which depends only on the topology of $M^2$. We have proved elsewhere various generalisations of this result which apply to higher dimensional simplicial complexes $K^n$: see [18], [19] and [20].

In this paper we turn things around and show that if a simplicial complex $K^n$ can be suitably colored by not too many colors, then it p.l. embeds in a given $\mathbb{R}^m$. As typical specimens of such results we have the following two:

**Theorem 2 (2.5.1).** Let $G(K^n_\delta)$ denote the graph whose vertices are pairs $(v, \theta)$ where $v$ is a vertex of $K^n$ and $\theta$ a maximal simplex of $K^n$ not containing $v$, with $(v_1, \theta_1)$ adjacent to $(v_2, \theta_2)$ iff $v_1 \in \theta_2$ and $v_2 \in \theta_1$. If $G(K^n_\delta)$ has chromatic number $\leq m + 1$ and $2m \geq 3(n + 1)$, or else $n - 1$ and $m - 2$, then $K^n$ p.l. embeds in $\mathbb{R}^m$.

**Theorem 6 (3.2.1).** Let $G(X^n)$ denote the graph whose vertices $X^n_i$ are closures of the non-singular edge-less components of the underlying polyhedron $X^n$ of $K^n$, with $X^n_i$ adjacent to $X^n_j$ iff $X^n_i$ is disjoint from $X^n_j$. If $G(X^n)$ is bichromatic and $n \neq 2$ then $K^n$ p.l. embeds in $\mathbb{R}^{2n}$.

Note that Theorem 6 above includes the well known fact that an $n$-pseudo-manifold p.l. embeds in $\mathbb{R}^{2n}$. The hypotheses of this theorem are relaxed considerably in Theorem 8 (3.4.2) whose statement involves some equivariant cohomology.

In Theorem 2 above, $K^n_\delta$ denotes a self-dual poset, the dual deleted product, which we associate canonically to each simplicial complex $K^n$. Theorems 3 and 4 of (2.5) are analogues of Theorem 2 for graphs $G(K^n_\delta)$ arising out of some sub self-dual posets $K^n_\delta$ of $K^n$. Theorem 3 is in fact a common generalization of Theorem 2 above and the Lovász-Kneser Theorem [12] which appears in this setting only as a very special colorability implies embeddability.
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We thus obtain, besides the classical Kneser graphs, a host of new examples of highly chromatic graphs which are triangle-free (and much more).

1.2. Method of proof.

There are two main ingredients involved in our proofs:

The first of these is the idea, going back at least to van Kampen [25], 1932, of using the deleted product \( K^n_\alpha \) of a simplicial complex \( K^n \). Recall that \( K^n_\alpha \) is the cell complex consisting of all cells of the type \( \sigma \times \theta \) where \( \sigma \) and \( \theta \) are disjoint simplices of \( K^n \). One equips \( K^n_\alpha \) with the involution \( (x, y) \rightarrow (y, x) \). The deleted product is important in Embedding Theory because embeddings \( f : K^n \rightarrow \mathbb{R}^m \) yield \( \mathbb{Z}_2 \)-maps \( f_\alpha : K^n_\alpha \rightarrow S^{m-1} \) by

\[
f_\alpha(x, y) = \frac{f(x) - f(y)}{|f(x) - f(y)|}.
\]

Conversely, Weber's Theorem [28], 1967, tells us that for \( 2m \geq 3(n + 1) \) the existence of such an \( f_\alpha \) implies the existence of an embedding \( f \).

The second idea came to the fore with Lovász's proof [12], 1978, of Kneser's Conjecture [11], 1955, viz. the idea of using a coloring to construct a suitable \( \mathbb{Z}_2 \)-map. We do not however use Lovász's "neighborhood complexes" and instead, taking our cue from a subsequent paper [13] of Lovász on "strongly self-dual polytopes", find it convenient to work with self-dual posets, i.e., finite partially ordered sets equipped with non-degenerate order reversing involutions. This idea ties up with deleted products because the latter are self-dual posets provided one uses the opposite order on the second factor.

1.3. Summary.

In (2.1) to (2.3) we review the basic definitions. Then we prove Theorem 1 (2.4.2): The graph of a self-dual poset has chromatic number \( \leq k + 2 \) only if there is a \( \mathbb{Z}_2 \)-map from its space to \( S^k \). In (2.5) we prove the aforementioned Theorems 2, 3 and 4 pertaining to the self-dual poset \( K^n_\alpha \) and its subposets \( K^n_\lambda \). In particular we obtain a new proof for the Lovász-Kneser Theorem. Then we show in (3.1) that an "obvious" generalization of the classical Kneser graphs is not of much interest. In (3.2) we give a direct proof of Theorem 6 by using the van Kampen cone construction.\(^1\) In (3.3) we review the equivariant cohomology of fixed point free \( \mathbb{Z}_2 \)-spaces. This is used in (3.4.1) to obtain some corollaries of Weber's Theorem, e.g., that for \( n \neq 2 \) a polyhedron \( X^n \) unknots in \( \mathbb{R}^{2n-1} \) only if it embeds in \( \mathbb{R}^n \). Next we state and prove Theorem 8,

\(^1\) This is one of the two main constructions needed in Embedding Theory. The other is the Whitney Trick [29].
the aforementioned stronger form of Theorem 6, using Weber's Theorem. To do this we introduce a certain interesting sub $Z_2$-polyhedron of the deleted product which we call the \textit{Kneser deleted product} (3.4.2). For some combinatorial purposes \textit{deleted joins} $K_n^d$ — which go back at least to Flores [4], 1933 — are more convenient than deleted products $K_n^d$. We use this idea in (3.5) to give a polynomial bound for the codimension one least valence of $K^n$ in terms of the "extent" to which $K^n$ can p.l. knot in $\mathbb{R}^{2n+1}$. We conclude by pointing out some evidence which tends to support the conjectures made in [19].

Since we are going to stick to the piecewise linear category, \textit{the adjective p.l. is quite often omitted}. Unless otherwise stated all spaces, maps, embeddings, homeomorphisms, etc., should be assumed p.l. (The main exception will be the use of some projection maps. These too can be replaced by p.l. maps by means of the standard way of getting around "the standard mistake"; e.g., see [10], pp. 20–21.)

2. Self-dual posets

(2.1) \textit{Posets, complexes, spaces.}

(2.1.1) We will deal mainly with the following categories.

(A) Finite partially ordered sets or \textit{POSETS} and functions between them which are \textit{MONOTONE}, i.e., order preserving or order reversing.

(B) Finite abstract simplicial \textit{COMPLEXES} and functions between them which are \textit{SIMPLICIAL}.

(C) Piecewise linear \textit{SPACES} and \textit{MAPS} between them which are piecewise linear.

The isomorphisms of category (C) are called piecewise linear \textit{homeomorphisms}. We say that $X_1$ \textit{embeds} piecewise linearly in an $X_2$ if $X_1$ is homeomorphic to a closed subspace $\bar{X}_1$ of $X_2$.

(2.1.2) Also we will need the corresponding equivariant categories for the group $Z_2$:

A $Z_2$-\textit{object} is an object equipped with an involutive morphism and between $Z_2$-objects one considers $Z_2$-\textit{morphisms}, i.e., those which commute with the involutions.

(2.2) \textit{Space of a poset.}

(2.2.1) We have the following functors between the above categories.

(B) $\Downarrow$ (A). Each simplicial complex is considered as a poset under the partial order $\subseteq$. This makes (B) into a subcategory of (A).

(A) $\Rightarrow$ (B). The \textit{derived} functor associates to each poset $P$ the simplicial complex $P'$ of \textit{chains} (i.e., totally ordered subsets) of $P$.

\footnote{Our simplices will be non-empty finite sets.}
(B) \rightarrow (C). This functor associates to each abstract simplicial complex \( K \) the piecewise linear space \( |K| \) underlying the corresponding geometrical simplicial complex obtained by thinking of each vertex \( v \) as the \( v \)-th unit vector of the vector space \( \mathbb{R}^{\text{vert}} K \).

A simplicial complex \( K \) is called a piecewise linear triangulation of the compact piecewise linear space \( X \) if \( |K| \) is p.l. homeomorphic to \( X \).

(2.2.2) A simplicial complex \( K \) embeds rectilinearly in a real vector space \( V \) if one has a linear map

\[
\mathbb{R}^{\text{vert}} K \rightarrow V
\]

which is one-one on \( |K| \).

It is known that each compact piecewise linear space \( X \) occurs as the image \( \eta([K]) \) of some such rectilinear embedding. (Such a \( K \) may be called a rectilinear triangulation of \( X \).) Likewise each piecewise linear map \( f: X_1 \rightarrow X_2 \) between compact spaces can be written as \( \eta_1 \circ \eta \circ \eta_1^{-1} \) for suitable choices of \( K_1, K_2, \eta_1, \eta_2, \) and simplicial map \( \eta: K_1 \rightarrow K_2 \).

Examples of Cairns [3] and van Kampen [26] show that the p.l. embeddability of a \( |K| \) in \( V \) does not guarantee the rectilinear embeddability of \( K \) in \( V \).

(2.2.3) Any homeomorph of \( |P'| \) will be denoted by \( X_P \) and said to be the space of the poset \( P \). Note that for any simplicial complex \( K, K' \) can be identified with the first barycentric derived of \( K \) and so \( |K'| \) is homeomorphic to \( |K| \). Hence the space of any poset is homeomorphic to that of its derived.

By a cell we understand the convex hull of a non-empty finite subset of a vector space \( V \). A finite set \( P \) of cells of \( V \) constitutes a cell complex if (1) the relative interiors of any two distinct cells are disjoint and (2) the relative boundary of any \( i \)-dimensional cell \( \sigma, \ i \geq 0, \) is a union of cells of dimensions \(< i \); these, and \( \sigma \) itself, are called the faces of \( \sigma \). We make \( P \) into a poset by letting \( \theta \leq \sigma \) if \( \theta \) is a face of \( \sigma \). If \( P \) is a cell complex \( P' \) can once again be identified with the simplicial complex arising as the first barycentric derived of \( P \) and so the space of the poset \( P \) is homeomorphic to the subspace \( |P'| \) of \( V \) covered by the cell complex \( P \).

Note that a cell complex is a geometrical simplicial complex iff it is isomorphic as a poset to an abstract simplicial complex.

(2.3) Deleted products, Weber’s Theorem.

(2.3.1) The cartesian product \( X \times X \) of a space with itself shall be equipped with the involution \( s \) which switches the coordinates: \( s(x_1, x_2) = (x_2, x_1) \). The deleted product \( X_\# \) of \( X \) is the \( \mathbb{Z}_2 \)-subspace of \( X \times X \) consisting of all pairs \( (x_1, x_2), x_1 \neq x_2 \). Each embedding \( \varphi: X \rightarrow Y \) induces a \( \mathbb{Z}_2 \)-map \( \varphi_\#: X_\# \rightarrow Y_\# \) by

\[
\varphi_\#(x_1, x_2) = (\varphi(x_1), \varphi(x_2)).
\]
Note that if an embedding \( \varphi_0 \) is isotopic to \( \varphi_1 \) via the embeddings \( \varphi_t \), \( 0 \leq t \leq 1 \), then \( \varphi_{0*} \) is \( \mathbb{Z}_2 \)-homotopic to \( \varphi_{1*} \) via the \( \mathbb{Z}_2 \)-maps \( \varphi_{t*} \), \( 0 \leq t \leq 1 \).

Any space p.l. homeomorphic to an \( m \)-dimensional cell (resp. its boundary) is called an \( m \)-ball \( B^m \) (resp. \( (m - 1) \)-sphere \( S^{m - 1} \)). The 0-sphere \( S^0 \) consists of 2 points and has the involution which interchanges the 2 points. We will equip \( S^k \), the \( (k + 1) \)-fold join of \( S^0 \), with the antipodal involution, i.e., the \( (k + 1) \)-fold join of the involution of \( S^0 \).

**Lemma 1.** There is a strong \( \mathbb{Z}_2 \)-deformation of the deleted product of an \( m \)-dimensional Euclidean space, or of an \( m \)-ball, onto an \( (m - 1) \)-sphere.

**Proof.** Consider the orthogonal projection of \( \mathbb{R}^n \times \mathbb{R}^m \) onto the orthogonal complement

\[
\Delta^+ = \{(v, -v) : v \in \mathbb{R}^n\}
\]

of the diagonal subspace

\[
\Delta = \{(v, v) : v \in \mathbb{R}^m\}.
\]

Thus each isotopy

\[
X \xrightarrow{\varphi} \mathbb{R}^n
\]

determines a \( \mathbb{Z}_2 \)-homotopy

\[
X \xrightarrow{\varphi_{0*}} S^{m - 1}.
\]

(2.3.2) The following result illustrates the importance of the deleted product functor.

**Weber's Classification Theorem.** If \( 2m > 3(n + 1) \) (resp. \( 2m = 3(n + 1) \)), and \( X^n \) is compact, then \( \varphi \mapsto \varphi_* \) sets up a bijective (resp. surjective) correspondence between isotopy classes of embeddings of \( X^n \) in \( \mathbb{R}^m \) and \( \mathbb{Z}_2 \)-homotopy classes of \( \mathbb{Z}_2 \)-maps \( X^n \to S^{m - 1} \).

This theorem is due to Weber [28], 1967; the surjectivity of \( \varphi \mapsto \varphi_* \) in the special case \( m = 2n, n \geq 3 \), was conjectured (and partly proved) by van Kampen [25], 1932, and proved, independently, by Wu [31], 1956, and Shapiro [22], 1957. A theorem exactly analogous to Weber's is valid for smooth embeddings of smooth \( n \)-manifolds in \( \mathbb{R}^m \) and was proved by Haefliger [8], 1962.

(2.3.2a) The results of van Kampen-Wu-Shapiro were stated differently in terms of the vanishing of some cohomology classes (see (3.4.1) below). It was Haefliger who reformulated this vanishing condition into an equivalent one.
involving the existence of a \( \mathbb{Z}_2 \)-map: this quickly led to the aforementioned stronger results of Haefliger and Weber. We note the following addendum to Weber’s Theorem for the case \( n = 1, m = 2 \).

A Graph Planarity Criterion. A graph \( X^1 \) is planar iff there is a \( \mathbb{Z}_2 \)-map from \( X^1 \) to \( S^1 \).

This surprisingly little known result must have been known to van Kampen by 1932, but the only place where I could find it explicitly stated (in its cohomological formulation) is p. 210 of Wu’s book [32]. One can prove it directly without defining the van Kampen Obstruction as follows.

It is easily seen that there is a \( \mathbb{Z}_2 \)-map from \( X^1 \) to \( S^1 \) iff there is a \( \mathbb{Z}_2 \)-map from the deleted join (see (3.5.1) below) \( X^1_\bullet \) to \( S^2 \). But the deleted joins of the Kuratowski Graphs are \( \mathbb{Z}_2 \)-homeomorphic to \( S^3 \) (see (3.5.2)). Hence by the Borsuk-Ulam Theorem, \( X^1 \) cannot have a subspace homeomorphic to a Kuratowski Graph, and so must be a planar graph.

We will show elsewhere that one can turn things around and prove this criterion without using Kuratowski’s Theorem, and then deduce the latter from it. By using Weber’s Theorem such techniques also yield some higher dimensional generalizations of Kuratowski’s Theorem: see [34].

(2.3.3) For any poset \( P \) (and in particular for simplicial complexes) we equip \( P \times P \) with the product partial order, \((a_1, a_2) \leq (b_1, b_2) \) iff \( a_1 \leq b_1, a_2 \leq b_2 \), and the involution \( * \) which switches the coordinates, \( *(a_1, a_2) = (a_2, a_1) \). If \( K \) is a simplicial complex its deleted product \( K_\# \) will be the \( \mathbb{Z}_2 \)-subposet of \( K \times K \) consisting of all pairs \((\sigma_1, \sigma_2) \), \( \sigma_1 \cap \sigma_2 = \emptyset \). Note that \( K_\# \) can be considered as a \( \mathbb{Z}_2 \)-cell complex covering a compact portion of the deleted product of the space \( X_K \). In fact one can say more.

Proposition 1. For any simplicial complex \( K \), \( X_K \) is a strong \( \mathbb{Z}_2 \)-deformation retract of \((X_K)_\# \).

Thus the \( \mathbb{Z}_2 \)-homotopy type of the space of \( K_\# \) depends only on the homeomorphism type of the space of \( K \). This result is due to Wu [30]. (Or else see [32] or [9]. Note that the argument on p. 257 of [22] is flawed because \( \beta(p, q) \) does not vary continuously with \( p \) and \( q \).

(2.4) Self-dual posets.

(2.4.1) A \( \mathbb{Z}_2 \)-poset \( (P, \nu) \) is called a self-dual poset if its involution \( \nu \) is order reversing and non-degenerate, i.e., if \( a \not\leq \nu(a) \) for \( a \in P \).

The \( \nu \)-product \( P_\# \) of a self-dual poset \( (P, \nu) \) is the \( \mathbb{Z}_2 \)-subposet of \( P \times P \) consisting of all pairs \((a, b) \), \( a \leq \nu(b) \).

\footnote{In this context we will denote the cells of \( K_\# \) by \( \sigma_1, \sigma_2 \) rather than by \((\sigma_1, \sigma_2)\).}
Lemma 2. For any self-dual poset $(P, v)$, $X_P$ and $X_{P'}$ are $\mathbb{Z}_2$-homotopy equivalent.

Proof. Define $\theta: P'_0 \rightarrow P'$ by associating to each chain $(a_1, b_1) < \cdots < (a_n, b_n)$ of $P$, the chain $a_1 \leq \cdots \leq a_n \leq v(b_n) \leq \cdots \leq v(b_1)$ of $P$. Though it is not simplicial $\theta$ is surjective, inclusion preserving and commutes with the simplicial involutions $s': P'_0 \rightarrow P'_0$ and $v': P' \rightarrow P'$. For each simplex

$$\sigma^m = \{ c_0 < c_1 < \cdots < c_m \}$$

of $P'$, the subcomplex $\sigma^m_0$ of all faces of $\sigma^m$ has as its pullback the subcomplex $\theta^{-1}(\sigma^m_0)$ of $P'_0$ consisting of all chains of the form

$$\{ (c_{i_0}, v(c_{j_0})) < \cdots < (c_{i_r}, v(c_{j_r})) \}$$

where $0 \leq i_0 \leq \cdots \leq i_r \leq j_0 \leq \cdots \leq j_r \leq m$. Since $\theta^{-1}(\sigma^m_0)$ is a cone over the vertex $(c_0, v(c_0))$ its space is contractible. Thus a $\mathbb{Z}_2$-section $\Phi: X_{P'} \rightarrow X_{P'_0}$ of the $\mathbb{Z}_2$-map $|\theta'|$: $X_{P'_0} \rightarrow X_{P'}$, can be constructed by an upward induction on the $m$-skeletons $(X_{P'_0})^m = \bigcup_m X_{P'_0}$. This $\Phi$ is the requisite homotopy inverse of $|\theta'|$.

(2.4.2) The graph of a self-dual poset $G(P, v)$ is the graph (i.e., 1-dimensional simplicial complex) whose vertices are the minimal elements of $P$ with 2 vertices $a$ and $b$ joined iff $a < v(b)$. The chromatic number of this graph will also be referred to as the chromatic number of the self dual poset.

Theorem 1. A self-dual poset $P$ has chromatic number $\leq k + 2$ only if there is a $\mathbb{Z}_2$-map from $X_P$ to $S^k$. Thus if the space $X_P$ of a self-dual poset $P$ is $k$-connected then its chromatic number is at least $k + 3$.

Proof. Let $P_0$ denote the subset of $P$ consisting of the minimal elements and let $\varphi: P_0 \rightarrow \{1, 2, \ldots, k + 2\}$ be a function such that $a \leq v(b)$ implies $\varphi(a) \neq \varphi(b)$. Let $\sigma^k_{k+1}$ denote the simplicial complex whose simplices are all the non-empty subsets of $\{1, 2, \ldots, k + 2\}$. We can define a $\mathbb{Z}_2$-monotone function

$$f: P_0 \rightarrow (\sigma^{k+1}_{k+1})_*$$

by

$$(a, b) \mapsto (\varphi(A), \varphi(B))$$

where $A$ (resp. $B$) denotes the subset of all minimal elements $c \leq a$ (resp. $c \leq b$). This follows because if $a_0 \in A, b_0 \in B$, then $a_0 \leq a \leq v(b) \leq v(b_0)$ and so $\varphi(a_0) \neq \varphi(b_0)$: thus $\varphi(A) \cap \varphi(B) = \emptyset$. Since $\sigma^{k+1}_{k+1}$ triangulates a $(k + 1)$-disk its deleted product has, by Proposition 1, the $\mathbb{Z}_2$-homotopy type
of the deleted product of a \((k + 1)\)-disk, and so, by Lemma 1, that of the 
k-sphere \(S^k\). (In fact it is easy to see directly that \(\{(a_{k+1})_n\}\) is \(Z_2\)-homeomorphic to \(S^k\).) On the other hand Lemma 2 gives us a \(Z_2\)-map from \(X_p\) to \(X_{r_p}\). Thus we get a \(Z_2\)-map from \(X_n\) to \(S^k\).

To see the second part note that under the given connectedness hypothesis one has a \(Z_2\)-map \(S^{k+1} \to X_p\): for each \(0 \leq r \leq k\) suitably extend the \(Z_2\)-map \(S' \to X_p\) from the equator \(S' \to X_p\) to \(S^r\) to \(S^{r+1}\) to the northern and southern hemispheres. If chromatic number were less than \(k + 3\) we would have a \(Z_2\)-map \(S^{k+1} \to S^k\). This contradicts Borsuk's Theorem [2].

The following cases of Theorem 1 are due to Lovász [12], [13], and Walker [27]: (1) \(P\) is a "strongly self-dual polytope"; (2) \(P\) is "the proper part \(\widetilde{L}(G)\) of the ortholattice \(L(G)\) of a graph \(G\)"; such an \(\widetilde{L}(G)\) has the same homotopy type as \(\mathcal{N}(G)\), "the neighborhood complex of the graph \(G\).

\[(2.5) \quad \text{Dual deleted product.}\]
\[(2.5.1) \quad \text{If we equip the set}\]
\[
\{(a_{\sigma_1}, a_{\sigma_2}) : a_{\sigma_1} \in K, a_{\sigma_2} \in K, a_{\sigma_1} \cap a_{\sigma_2} = \emptyset\}
\]
of all ordered pairs of disjoint simplices of \(K\) with the involution \(s\) and the partial order \((a_{\sigma_1}, a_{\sigma_2}) \leq (a_{\sigma_1}', a_{\sigma_2}')\) iff \(a_{\sigma_1} \subseteq a_{\sigma_1}'\) and \(a_{\sigma_2} \subseteq a_{\sigma_2}'\) (instead of the product partial order of (2.3.3)) then we get a self-dual poset \(K_{c\sigma}\) which will be called the dual deleted product of \(K\).

**Theorem 2.** If \(G(K_{c\sigma})\) has chromatic number \(\leq m + 1\), (here \(m > 0\)), then there is a \(Z_2\)-map \((X_{K_{c\sigma}})_n \to S^{m-1}\). Thus if further \(2m \geq 3(n + 1)\), or else \(n = 1\) and \(m = 2\), then \(X_{K_{c\sigma}}\) embeds in \(R^m\).

**Proof.** The second part will follow from Weber's Theorem. To prove the first part we note that by Proposition 1 it suffices to find a \(Z_2\)-map

\[X_{K_{c\sigma}} \to S^{m-1}.
\]
Under the given chromatic hypothesis Theorem 1 supplies us with a \(Z_2\)-map

\[X_{K_{\sigma}} \to S^{m-1}.
\]
Thus the result follows from the following lemma.

**Lemma 3.** For any simplicial complex \(K\), \(X_{K_{\sigma}}\) has the same \(Z_2\)-homotopy type as \(X_{K_{\sigma}}\).

**Proof.** Let \(K = \{|K|\}\) and identify \(K\) with the geometrical simplicial complex covering \(X\) and thus \(K_{\sigma} \subset K \times K\) with the cell complex consisting of all
cells $\sigma \times \theta, \sigma \in K, \theta \in K, \sigma \cap \theta = \emptyset$. The space $|K_\alpha|$ covered by $K_\alpha$ is thus a compact $\mathbb{Z}_2$-subspace $Y$ of the deleted product, $K_\alpha \times X = |K| \times |K|$, of $X$.

Choose a barycenter $\hat{\theta}$ in each $\sigma \in K$ and think of $K'$ as the barycentric derived of $K$, i.e., each $\sigma' \in K'$ which is a chain $\sigma_0 \subset \cdots \subset \sigma_k$ of $K$, is thought of as the geometrical simplex which is the convex hull $[\hat{\sigma}_0, \ldots, \hat{\sigma}_k]$. Let us now cut up the cells $\sigma \times \theta$ of $K \times K$ into smaller cells of the form $\sigma' \times \theta', \sigma' \subset K', \theta' \subset K'$, $\sigma' \subseteq \sigma$, $\theta' \subseteq \theta$. This gives us the cell complex $K' \times K'$; we denote by $C$ the subcomplex of $K' \times K'$ covering $|K_\alpha|$.

Consider the cells $\sigma' \times \theta' \subseteq C$,

$$
\sigma' = [\hat{\sigma}_0, \ldots, \hat{\sigma}_k], \quad \sigma_0 \subseteq \cdots \subseteq \sigma_k,
\theta' = [\hat{\theta}_0, \ldots, \hat{\theta}_l], \quad \theta_0 \subseteq \cdots \subseteq \theta_l.
$$

We can further cut them up into simplices

$$
\left[ (\hat{\sigma}_{i_0}, \hat{\theta}_{j_0}), \ldots, (\hat{\sigma}_{i_k}, \hat{\theta}_{j_k}) \right], \quad 0 \leq i_0 \leq \cdots \leq i_k, 0 \leq j_0 \leq \cdots \leq j_l.
$$

This gives us a simplicial complex isomorphic to $K'_\alpha$. Alternatively we can cut them up into the simplices

$$
\left[ (\hat{\sigma}_{i_0}, \hat{\theta}_{j_0}), \ldots, (\hat{\sigma}_{i_k}, \hat{\theta}_{j_k}) \right], \quad 0 \leq i_0 \leq \cdots \leq i_k, 0 \leq j_0 \leq \cdots \leq j_l, 0 \leq j_0 \leq \cdots \leq j_l.
$$

This gives us a simplicial complex isomorphic to a subcomplex of $K'_\alpha$. Thus $|K'_\alpha| = Y$ and $|K'_\alpha| \supseteq Y$ (this already suffices to complete the proof of Theorem 2). Figure 1 shows an example where $|K'_\alpha|$ is not homeomorphic to $Y$.

To get an isomorph $K'_\alpha$ of $K'_\alpha$ we have thus to consider also the simplices

$$
\left[ (\hat{\sigma}_{i_0}, \hat{\theta}_{j_0}), \ldots, (\hat{\sigma}_{i_k}, \hat{\theta}_{j_k}) \right],
$$

$0 \leq i_0 \leq \cdots \leq i_k, \sigma_0 \cap \theta_j = \emptyset, \sigma_j \cap \theta_j = \emptyset$, arising in the above fashion, within cells $\sigma \times \theta$ of $K \times K$ which are not in $K_\alpha$. Since $\sigma \cap \theta \neq \emptyset$, note that any such simplex of $K'_\alpha$ cannot have the barycenter $(\hat{\sigma}, \hat{\theta})$ of $(\sigma, \theta)$ as a vertex. For each cell $\sigma \times \theta$ of $(K \times K) - K_\alpha$ let $r_{\sigma \times \theta}$ denote the radial deformation of $(\sigma \times \theta) - (\hat{\sigma}, \hat{\theta})$ from the barycenter $(\hat{\sigma}, \hat{\theta})$ towards the boundary $\partial(\sigma \times \theta)$. This deformation $r_{\sigma \times \theta}$ maps $|K'_\alpha| \cap |\sigma \times \theta|$ into itself. To see this we note that a simplex of above type lies outside $\partial(\sigma \times \theta)$ iff $\hat{\theta}_j = \theta$ and $\sigma_j = \sigma$ and that under the deformation it moves over the region given as the convex hull of the vertices of the simplex and the points $(\hat{\sigma}_{i_0}, \hat{\theta}_{j_0}), 0 \leq k < l \leq r, \sigma_{i_{k+1}} = \theta, \sigma_{j_{l+1}} = \sigma$ (cf. shaded areas in Figure 1). Since $\sigma_{i_k} \times \theta_{j_l} \in K_\alpha$, it is easy to check that this region is covered by simplices of $K'_\alpha$. We can now obtain a $\mathbb{Z}_2$-deformation of $|K'_\alpha|$ onto $|K_\alpha|$ by a step-by-step
procedure in which at each step these radial deformations are used in those highest dimensional antipodal pair of cells \( \{ \sigma \times \theta, \theta \times \sigma \} \) of \( K \times K - K_+ \) which still contain some part of the deformed \( |K_+| \).

(2.5.2) Remarks.
(a) Twisted triangulations analogous to \( K_+ \) can be defined also on higher “deleted powers” and “configuration complexes” of \( K \). We will show elsewhere that this leads to interesting generalizations of the results of this paper for finite groups \( G \) other than \( \mathbb{Z}_2 \).
(b) Note that the argument used to prove Lemma 3 is applicable to any \( \mathbb{Z}_2 \)-subcomplex of \( K \times K \). Thus one has:

**Lemma 3’.** If a \( \mathbb{Z}_2 \)-subcomplex \( E \) of the cell complex \( K \times K \) is considered as a poset \( P \) under \( \preceq \), then \( X_P \) has the same \( \mathbb{Z}_2 \)-homotopy type as \( X_E \).

This observation will be used in the proof of Theorem 3 below.
(c) In the above proof of Theorem 2 one can avoid using Theorem 1 by means of the following direct construction.

Let \( S \subset K_\Sigma \) denote the set of all minimal elements. Let

\[
\sigma : S \to \{0, 1, \ldots, m\} = \sigma^m
\]

denote the given coloring. So

\[
(v_1, \theta_1), (v_2, \theta_2) \in S, v_1 \in \theta_2, v_2 \in \theta_1 \Rightarrow f(v_1, \theta_1) \neq f(v_2, \theta_2).
\] (1)
We now define a (non-simplicial) function $F: K_\Box \rightarrow \sigma_0^m$ (the simplicial complex containing all faces of $\sigma^m$) by

$$F(\alpha, \beta) = \{ f(\nu, \theta) \mid (\nu, \theta) \in S, \nu \in \alpha, \theta \supseteq \beta \}. \quad (2)$$

Note that $F(\alpha, \beta)$ is indeed always a non-empty subset of $\sigma^m$. (Had we started from a coloring $f$ defined over some other $S \subset K_\Box$ we would have got an $F$ with a possibly smaller domain $S \subset K_\Box$.) Next we note that $F(\alpha, \beta)$ is always disjoint from $F(\beta, \alpha)$ because

$$v_1 \in \alpha, \theta_1 \supseteq \beta, v_2 \in \beta, \theta_2 \supseteq \alpha \Rightarrow v_1 \in \theta_2, v_2 \in \theta_1 \Rightarrow f(v_1, \theta_1) \neq f(v_2, \theta_2)$$

by (1). Finally note that

$$\alpha_1 \subseteq \alpha_2, \beta_1 \supseteq \beta_2 \Rightarrow F(\alpha_1, \beta_1) \subseteq F(\alpha_2, \beta_2).$$

Thus we can define a simplicial map $f_\Box: K_\Box \rightarrow (\sigma_0^m)_\Box$ by mapping each simplex

$$\{(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, \alpha_1 \subseteq \alpha_2 \subseteq \ldots, \beta_1 \supseteq \beta_2 \supseteq \ldots \}$$

of $K_\Box$ to the simplex

$$\{(F(\alpha_1, \beta_1), F(\beta_1, \alpha_1)), (F(\alpha_2, \beta_2), F(\beta_2, \alpha_2)), \ldots \}$$

of $(\sigma_0^m)_\Box$. Then $|f_\Box|$ is the requisite $Z_2$-map from $X_{K_\Box}$ to $S^{m-1}$.

(d) Even the graph-theoretical case $n = 1, m = 2$, of Theorem 2 seems to be new. It says that for any non-planar graph $K^1$, the associated graph $G(K^1_\Box)$ must have chromatic number bigger than 3. In the opposite direction we have:

**A FOUR COLOR THEOREM.** For any planar graph $K^1$ the associated graph $G(K^1_\Box)$ has chromatic number $\leq 4$.

We will show elsewhere that it is enough to consider the case when $K^1$ is hamiltonian. Let $L^1 \subset K^1$ be a circle containing all the vertices of $K^1$. Choose an embedding of $K^1$ in $\mathbb{R}^2$ and let the vertices of $K^1$ be $1, 2, \ldots, t$, as one proceeds along $L^1$ in a clockwise direction. A vertex $(\{a\}, \{b, c\})$, $b < c$, of $G(K^1_\Box)$ will be given (i) the color $\alpha$ if $a < b < c$, (ii) the color $\beta$ if $b < a < c$ and $(b, c)$ is either $\{1, t\}$ or an edge lying in the bounded component of $\mathbb{R}^2 - |L^1|$, (iii) the color $\gamma$ if $b < a < c$ and $(b, c)$ is in the unbounded component of $\mathbb{R}^2 - |L^1|$, and (iv) the color $\delta$ if $b < c < a$. It is easily verified that this is a good 4-coloring of the vertices of $G(K^1_\Box)$. 
I do not know whether the above bound is the best possible.

Also one has Heawood-type upper bounds for the chromatic number of $G(K_{1,3}^N)$ as $K^1$ runs over all graphs embeddable in any fixed 2-pseudomanifold $X^2$.

(e) It is easy to check that for any graph $K^1$ the associated graph $G(K_{1,3}^{K^1})$ does not contain a complete graph on four vertices. However there are graphs $K^1$ for which the chromatic number of $G(K_{1,3}^{K^1})$ can be arbitrarily big. For example let $\sigma^N = \{0, 1, \ldots, N\}$ and let $K^1 = \sigma^N_0$, the complete graph on $N + 1$ vertices. Suppose there is a good vertex coloring of $G((\sigma^N_0)_{1,3})$ with $m$ colors. To any 2-simplex $\{a, b, c\} \subset \sigma^N$, $a < b < c$, assign the color of the vertex $(b, \{a, c\})$ of $G((\sigma^N_0)_{1,3})$. We note that any 3-simplex $\{e, f, g, h\} \subset \sigma^N$, $e < f < g < h$, has two incident 2-simplices, namely $\{e, f, g\}$ and $\{f, g, h\}$ which have different colors. By Ramsey's Theorem no such $m$-coloring of the 2-simplices of $\sigma^N$ is possible if $N$ is sufficiently big.

(2.5.3) Let $G_s(K_{1,3})$, $s \geq 0$, denote the graph whose vertices are pairs $(\sigma^s, \theta)$ where $\sigma^s$ is an $s$-simplex of $K$ and $\theta$ a maximal simplex of $K$ disjoint from $\sigma^s$, with $(\sigma^1_s, \theta_1)$ adjacent to $(\sigma^2_s, \theta_2)$ if $\sigma^1_s \subseteq \theta_2$ and $\sigma^2_s \subseteq \theta_1$. So $G_0(K_{1,3}) = G(K_{1,3})$. Note further that if a vertex $(\sigma^s, \theta)$ of $G_s(K_{1,3})$ has dim $\theta < s$, then it is an isolated vertex. Modulo such vertices $G_s(K_{1,3})$ coincides with $G(K_s)$, the graph of minimal elements of the subself-dual poset $K_s$ of $K_{1,3}$ consisting of all pairs $(\alpha, \beta)$ with dim $\alpha \geq s$, dim $\beta \geq s$. The following result generalizes Theorem 2.

**Theorem 3.** If $G_s(K_{1,3})$ has chromatic number $\leq m + 1 - 2s$ (here $m > 2s$), then there is a $Z_s$-map $X_{K_{1,3}} \rightarrow S^{m-1}$. Thus if further $2m \geq 3(n + 1)$ then $X_{K_{1,3}}$ embeds in $\mathbb{R}^m$.

**Proof.** Because of Theorem 2 we can assume that $K_s$ is a proper subset of $K_{1,3}$. Since $m - 1 - 2s \geq 0$, the given chromatic hypothesis and Theorem 1 (or else the construction of (2.5.2)(c)) supply us with a $Z_s$-map

$$|(K_{1,3}^s)| \rightarrow S^{m-1-2s}.$$

Any chain of $K_{1,3}^s$ is built up in a unique way from a chain of $K_{1,3}^s$ and a chain of the complementary self-dual poset $P_s = K_{1,3}^s - K_{1,3}$. We note that $|(K_{1,3}^s)|$ and $|P_s|$ are disjoint subspaces of $|(K_{1,3}^s)|$ and that any point $x$ of $|(K_{1,3}^s)|$ which is in neither of these subspaces is an interior point of a unique geometrical simplex of $(K_{1,3}^s)$ having some vertices in $(K_{1,3}^s)$ and some in $P_s$. In other words $|(K_{1,3}^s)|$ is a $Z_s$-subspace of the join $|(K_{1,3}^s)| \ast |P_s|$. Hence if we could produce a $Z_s$-map $g: |P_s| \rightarrow S^{2s-1}$, then we would have the desired $Z_s$-map

$$X_{K_{1,3}^s} = |(K_{1,3}^s)| \rightarrow S^{m-1-2s} \ast S^{2s-1} \equiv S^{m-1}.$$
Note that the cells $\sigma \times \theta \in P$ are all those cells of $K^n$ for which either $\dim \sigma$ or $\dim \theta$ is less than $s$. Thus they constitute a subcomplex $E$ of the cell complex $K$. By Lemma 3', $|P'|$ has the same $\mathbb{Z}_2$-homotopy type as its subspace $|E'|$. Thus there will be a $\mathbb{Z}_2$-map $g: |P'| \to S^{2s-1}$ iff such a $\mathbb{Z}_2$-map can be defined on the subspace $|E'|$. Let $T$ denote the full subcomplex of $E'$ spanned by vertices of the type $(\delta, \bar{\theta})$ where either $\dim \sigma \geq s$, $\dim \theta < s$ (the full subcomplex of $T$ determined by all such vertices will be called $T_1$) or $\dim \sigma < s$, $\dim \theta \geq s$ (these determine the subcomplex $T_2$ of $T$). We note that an edge of $T$ cannot have one vertex in $T_1$ and the other in $T_2$. So $T$ is the disjoint union of the antipodal subcomplexes $T_1$ and $T_2$. Thus we can find a $\mathbb{Z}_2$-map

$$|T| \xrightarrow{g_1} S^0.$$ 

The remaining vertices of $E'$ are of the type $(\delta, \theta)$, $\dim \sigma < s$, $\dim \theta < s$, and thus determine a full subcomplex $U$ of $E'$ of dimension $\leq 2(s-1)$. By working up inductively on the skeletons of $U$ we see that there is no obstruction to finding a $\mathbb{Z}_2$-map

$$|U| \xrightarrow{g_2} S^{2(s-1)}.$$ 

This gives us the required $\mathbb{Z}_2$-map

$$|E'| \xrightarrow{\delta_0 \oplus g_2} S^0 \ast S^{2(s-1)} \cong S^{2s-1}.$$ 

(2.5.4) Remarks. (a) The graphs $G_r(K^n_r)$ cannot contain a complete graph on more than

$$\left[ 1 + \frac{n+1}{s+1} \right]$$

vertices: If $(\sigma_i^r, \theta_j)$, $\ldots$, $(\sigma_l^r, \bar{\theta}_j)$ are all mutually adjacent, then $\sigma_i^r, \sigma_j^r, \ldots, \sigma_l^r$ would be mutually disjoint subsets of the simplex $\theta_j$. Since $\dim \theta_j \leq n$ we thus get $(t-1)(s+1) \leq n+1$ and so

$$t \leq \frac{n+1}{s+1} + 1.$$ 

(b) Theorem 3 gives a host of new examples of triangle-free graphs $G_r(K^n_r)$, $n \leq 2s$, $n-s$ large, having large chromatic numbers:

For instance if we make sure that the $n$-dimensional simplicial complex $K^n$ does not p.l. embed in $\mathbb{R}^{2s}$, then, by Theorem 3, the chromatic number of $G_r(K^n_r)$ will be at least $2(n-s) + 2$. Or, again, if $n$ is a power of 2, and $K^n$
is a triangulation of $\mathbb{R}P^n$, the real $n$-dimensional projective space, then the chromatic number of $G_s(K^n_2)$ is at least $2(n - s) + 1$. This follows because such a projective space does not p.l. embed in $\mathbb{R}^{n-1}$ (e.g., see Steenrod [24], p. 34, or Milnor [15], p. 120).

Since $n \leq 2s$ the high chromaticity of these graphs is obviously not due to any contained isomorphs of the “classical Kneser graphs”; see proof of Theorem 4 below.

(c) Conjecture. The chromatic number of the graph $G_{n-s}(K^n_2)$ is bounded as $K^n$ runs over all $n$-dimensional simplicial complexes embeddable in $\mathbb{R}^{n-1}$ and, more generally, as $K^n$ runs over all $n$-dimensional simplicial complexes embeddable in a fixed $X^{2^n}$ (cf. the analogous Conjecture 2 of [19] which deals with “Ramsey Colorings”, and (2.5.2)(d) above).

(2.5.5) We now take a closer look at the chromatic lower bounds which must be satisfied by the graphs $G_s(K^n_2)$ merely by virtue of the local topological fact that $K^n$ is $n$-dimensional. Since there is no $\mathbb{Z}_2$-map from $|K^n_2|$ to $S^{n-2}$, the following result is included in Theorem 3. We give below another argument which further clarifies the constructions introduced above.

**Theorem 4.** For any $n$-dimensional simplicial complex $K^n$ the chromatic number of $G_s(K^n_2)$ (here $n \geq 2s$) is greater than $n - 2s$.

**Proof.** For any subcomplex $L$ of $K$ one can find an isomorph of $G_s(L_0)$ in the graph $G_s(K_0)$: assign to each vertex $(\sigma^s, \theta)$ of $G_s(L_0)$ a vertex $(\sigma^s, \theta')$ of $G_s(K_0)$ where $\theta'$ is a maximal simplex of $K$ disjoint from $\sigma^s$ and containing $\theta$. Thus it is enough to prove the result when $K^n = a^n_s$, the simplicial complex consisting of all the faces of an $n$-simplex $a^n_s = \{0, 1, \ldots, n\}$. But $G_s(a^n_s)$ is obviously isomorphic to the $s$-th classical Kneser graph $G_s(a^n_s)$ of $a^n_s$, i.e., the graph whose vertices are the $s$-faces of $a^n_s$ with 2 vertices adjacent iff they are disjoint. The theorem follows because the well-known Kneser Conjecture [11], 1955, proved by Lovász [12], 1978, tells us that the chromatic number of this graph is exactly $n - 2s + 1$.

*The Lovász-Kneser Theorem can also be proved as follows.* That $n - 2s + 1$ colors suffice is very easy to see: assign to each $s$-face having a vertex in

$$\{0, 1, \ldots, n - 2s - 1\}$$

its first vertex and to all other $s$-faces the vertex $n - 2s$. Clearly $n - 2s = 0$ or 1 colors won’t do if $n = 2s$ or $n = 2s + 1$. So assume $n - 1 > 2s$. If $n - 2s$ colors would do, then Theorem 1 (or else the construction of (2.5.2)(c)) supplies us with a $\mathbb{Z}_2$-map

$$\left|(a^n_s)_{\Sigma}\right| \to S^{n-2s-2},$$
where \((\alpha^\alpha_n)_{\Sigma}\) is the subposet of \((\alpha^\alpha_n)_{\Sigma}\) consisting of all disjoint pairs of faces \((\alpha, \beta)\) of \(\alpha^\alpha_n\) with \(\dim \alpha \geq s\), \(\dim \beta \geq s\), i.e., all pairs \((\alpha, \beta)\) with

\[ s \leq \dim \alpha, \dim \beta \leq n - s - 1. \]

Consider also the subposet \((\alpha^\alpha_n)_{\Theta}\) consisting of all \((\alpha, \beta)\) with \(\dim \alpha + \dim \beta = n - 1\), i.e., all \((\alpha, \beta)\) for which \(\beta = \alpha^n - \alpha\). Any chain of \((\alpha^\alpha_n)_{\Theta}\) has at most \(n\) members (corresponding to \(0 \leq \dim \alpha \leq n - 1\)), out of which at most \(n - (n - 2s) = 2s\) (i.e., those with \(\dim \alpha\) not in \([s, n - s - 1]\)) are outside \((\alpha^\alpha_n)_{\Sigma}\). So (as in the proof of Theorem 3) we can construct a \(\mathbb{Z}_2\)-map from \(|(\alpha^\alpha_n)_{\Theta}|\) to \(S^{n-2s-2} \times S^{2s-1} \equiv S^{n-2}\). This is not possible because the poset \((\alpha^\alpha_n)_{\Theta}\) is isomorphic to the poset underlying the simplicial complex,
\( \sigma^n_{n-1} \), of all proper faces of \( \sigma^n \) (under \((\alpha, \beta) \mapsto \alpha\)), and so its space \(|(\sigma^n_{i})_{\sigma}| \) is homeomorphic to the \((n-1)\)-sphere \(|\sigma^n_{n-1}| \).

One can check either directly, or else using Lemmas 2 and 3, that the \(2(n-1)\)-dimensional \(\mathbb{Z}/2\) space \(|(\sigma^n_{i})_{\sigma}| \) has the same \(\mathbb{Z}/2\)-homotopy type as the \(\mathbb{Z}/2\)-subspaces \(|(\sigma^n_{i})_{\sigma}| \) and \(|(\sigma^n_{i})_{\sigma}| \), these being \(\mathbb{Z}/2\)-homeomorphs of \(S^{n-1}\). Figure 2 shows all these spaces and the deleted join \(|(\sigma^n_{i})_{\sigma}| \) (see (3.5.1)) for \(n = 2\).

3. Kneser graphs of polyhedra

(3.1) Interior of \(K_n\),

(3.1.1) Besides the generalisation \(G_r(K_n)\) — or even \(G(S)\), \(S \subset K_n\) — of the classical Kneser graphs, it is of interest also to examine some others. For example one can define the \(i\)-th Kneser graph \(G_i(K)\) of a simplicial complex \(K\) to have as vertices the \(i\)-simplices \(a^i\) of \(K\) with \(a^i_1 \) joined to \(a^i_2\) iff \(a^i_1 \cap a^i_2 = \emptyset\).

THEOREM 5. Let \(K^n\) be a homogeneously \(n\)-dimensional simplicial complex whose deleted product has the same \(\mathbb{Z}/2\)-homotopy type as its interior in \((X_{K^n})_0\). If \(G_n(K)\) has chromatic number \(\leq m + 1\), then there is a \(\mathbb{Z}/2\)-map \((X_{K^n})_0 \to S^{m-1}\). Thus if further \(2m \geq 3(n + 1)\) then \(X_{K^n}\) embeds in \(\mathbb{R}^m\).

Proof\(^4\). Let \(S_n(K)\) be the set of all \(n\)-simplices of \(K\) and let

\[ \varphi: S_n(K) \to \sigma^n = \{1, 2, \ldots, m + 1\} \]

be the postulated coloring; so \(a^i_1 \cap a^i_2 = \emptyset\) implies \(\varphi(a^i_1) \neq \varphi(a^i_2)\). Let \(P(K)\) denote the sub \(\mathbb{Z}/2\)-poset of \(K \times K\) consisting of all cells \(a_1 \times a_2 \subseteq \text{int} \{K_+\}\). (Since we are considering elements of \(K \times K\) as cells we write them as \(a_1 \times a_2\) instead of as pairs \((a_1, a_2)\). From the abstract viewpoint we have \(P(K) = \{(a_1, a_2): a_1 \in K, a_2 \in K, \bar{N}_K a_1 \cap \bar{N}_K a_2 = \emptyset\}\).) For each \(a_1 \times a_2 \in P(K)\), the sets \(\Sigma_1\) and \(\Sigma_2\) of all \(n\)-simplices incident to \(a_1\) and \(a_2\) respectively, are non-empty and each \(a^i_1\) in \(\Sigma_1\) is disjoint from each \(a^i_2\) in \(\Sigma_2\). Thus we can define an order reversing \(\mathbb{Z}/2\)-morphism

\[ f: P(K) \to (\sigma^n_m)_+ \]

by

\[ (a_1 \times a_2) \mapsto (\varphi(\Sigma_1) \times \varphi(\Sigma_2)) \].

\(^4\)See also [35] where a variant of this simple argument is used to establish a generalized Erdős-Kneser Conjecture.
In the barycentric subdivision \((K \times K)'\), the derived complex \((P(K))'\) occurs as the subcomplex made up of all simplices having vertices in \(\text{int}(K_n)\). Each point of \(|K \times K)|\) lying in neither \(|(P(K))'|\) nor in the complement of \(\text{int}(K_n)\) lies in a unique open simplex of \((K \times K)'\) having some vertices in \((P(K))'\) and some outside \(\text{int}(K_n)\); thus it is an interior point of a unique line segment having one end in \(|(P(K))'|\) and one outside \(\text{int}(K_n)\). By pushing along these line segments towards \(|(P(K))'|\) we see that \(|(P(K))'|\) is a \(Z_\gamma\)-deformation retract of \(\text{int}(K_n)\). Also, by hypothesis, \(|K_n|\) — which by Proposition 1 has the \(Z_\gamma\)-homotopy type of \((X_n)\) — has the same \(Z_\gamma\)-homotopy type as \(\text{int}(K_n)\), and we know that \(|(G_n)|\) has the \(Z_\gamma\)-homotopy type of \(S^{m-1}\). Thus \(f\) furnishes us with a \(Z_\gamma\)-map \((X_n) \to S^{m-1}\).

3.1.2 Remarks. (1) The condition \(|K_n| = \text{int}(K_n)\) used in the above theorem holds whenever \(K^n\) is an \(n\)-manifold, and its deleted product \(K^n\) a \(2n\)-manifold-with-boundary. However very few such \(K^n\)s will satisfy the chromatic hypothesis of Theorem 5.

(2) A colorability implies embeddability theorem of the above type cannot hold unconditionally. To see this consider the \(n\)-skeleton of a \((2n + 2)\)-simplex \(\sigma_n^{2n+2}\). It was proved by van Kampen [25] and Flores [4] that \(\sigma_n^{2n+2}\) does not embed in \(\mathbb{R}^{2n}\). On the other hand the Lovász-Kneser Theorem tells us that the chromatic number of \(G_n(\sigma_n^{2n+2})\) is only 3.

(3) However that is about as bad as things can be. Any \(K^n\) with \(G_n(\sigma_n^{2n+2})\) bichromatic embeds in \(\mathbb{R}^{2n}\). Indeed we will proceed now to show that there are graphs much smaller than \(G_n(\sigma_n^{2n+2})\), and depending only on the homeomorphism type of the underlying polyhedron \(X^n = |K^n|\), whose bichromaticity still forces the same conclusion.

3.2 The van Kampen construction.

3.2.4 A point \(x\) of a compact \(n\)-dimensional space \(X^n\), \(n \geq 1\), is called a singular point (or "of intrinsic dimension \(n - 1\" in the terminology of Akin [1]) if no triangulation of \(X^n\) contains \(x\) in the interior of an \(n\)-simplex. A singular point will be said to lie on the edge of \(X^n\) if, in some triangulation of \(X^n\), it is incident to exactly one \(n\)-simplex. Let \(\text{sing}(X^n)\) denote the subpolyhedron of all singular points, and \(X^n\) the closures, in \(X^n\), of the components of \(X^n - \text{sing}(X^n)\). We define \(G(X^n)\), the Kneser graph of polyhedron \(X^n\), to have as vertices all those \(X^n\) which have no points on the edge of \(X^n\), with vertices \(X^n\) and \(X^n\) joined iff \(X^n \cap X^n = \emptyset\).

A polyhedron \(X^n\) is said to unknot in \(\mathbb{R}^m\) if \(X^n\) embeds in \(\mathbb{R}^m\), and any two embeddings of \(X^n\) in \(\mathbb{R}^m\) are ambient isotopic to each other. Hudson's Isotopy Extension Theorem — see Hudson [10] for background and Akin [1], Corollary 17, p. 465 — assures us that, for \(m - n \geq 3\), this is equivalent to just demanding that any two embeddings of \(X^n\) in \(\mathbb{R}^m\) be isotopic to each other.

We now strengthen the theorem of Sarkaria [21] to the following.
Theorem 6. The graph \( G(X^n) \), \( n \neq 2 \), can be 2-colored only if \( X^n \) embeds in \( \mathbb{R}^{2n} \). And \( G(X^n) \), \( n \neq 1 \), can be 1-colored only if \( X^n \) unknotts in \( \mathbb{R}^{2n+1} \).

Proof. We’ll color the vertices of \( G(X^n) \) by \( w \) and \( b \). Let \( W \) be the union of all \( X^t_i \in G \) (i.e., those having a point on the edge of \( X^n \)), and all those \( X^m_t \in G \) which have color \( w \); and let \( B \) be the union of all \( X^m_t \in G \) having color \( b \). Since \( W \cap B \) has dimension \( \leq n-1 \) it embeds in the \( 2n-1 \) dimensional vector subspace
\[
\mathbb{R}^{2n-1} = \{ (x_1, x_2, \ldots, x_{2n}) : x_{2n} = 0 \}
\]
of \( \mathbb{R}^{2n} \), and we can extend this embedding to a general position map \( f : X^n \to \mathbb{R}^{2n} \) which images \( W - W \cap B \) (resp. \( B - W \cap B \)) into the half space \( x_{2n} < 0 \) (resp. \( x_{2n} > 0 \)). The map \( f \) is one-one except for a finite number of pairs of double points \( (p_1, p_2) \) which are non-singular.

If one of these double points lies in an \( X^t_i \in G \) we join it, via non-singular non-double points, to a point on the edge of \( X^n \) and delete from \( X^n \) an open regular neighborhood of this arc. The resulting polyhedron being homeomorphic to \( X^n \), this pair of double points gets eliminated.

Otherwise \( p_1 \in X^t_i \in G, p_2 \in X^t_j \in G \) with \( X^t_i \) and \( X^t_j \) having the same color, and so \( X^t_i \cap X^t_j \neq \emptyset \). Such a pair \( (p_1, p_2) \) is eliminated, when \( n \geq 3 \), by means of the van Kampen-Penrose-Whitehead-Teeman cone construction. (See [25], p. 152, and [16]; also [33], p. 66.) Briefly, as in [21], join \( p_1 \) to \( p_2 \) via an arc \( A \), all of whose (other) points are non-singular non-double points, with at most one exception, which is a singular point. Since \( n \geq 3 \), the circle \( C = f(A) \) bounds a 2-disk \( D \subseteq \mathbb{R}^{2n} \) meeting \( f(X^n) \) only in \( C \). Its regular neighborhood \( N(D) \) a 2-2n-disk meets \( f(X^n) \) in \( f(N(A)) \) with \( \partial N(D) \cap f(X^n) = f(\partial N(A)) \). Here \( N(A) \) denotes regular neighborhood of \( A \). From hypotheses on \( A, N(A) \) is a cone over its boundary \( \partial N(A) \). So we can alter \( f \) on \( N(A) \) by coning \( f(\partial N(A)) \) over an interior point of \( N(D) \).

The graph-theoretical case \( n = 1 \) is trivial. In fact, there is no \( \mathbb{X}^1 = |K^1| \), which satisfies the hypothesis and for which all vertices of \( K^1 \) have valence \( \geq 3 \): This follows by noting that the subcomplexes \( W^1 \) and \( B^1 \) of \( K^1 \) determined by the white and black edges of \( K^1 \) must in fact be in the closed stars of two vertices \( w \) and \( b \), but one cannot have \( K^1 = (S^1_k \cup w) \cup (S^1_k \cup b) \).

The second part follows by noting that if \( n \geq 2 \) and \( G(X^n) \) has no edge, then an analogous elimination of double points converts any general position map
\[
X^n \times [0, 1] \to \mathbb{R}^{2n+1} \times [0, 1]
\]
into a concordance between the pair of embeddings
\[
X^n \times \{0, 1\} \to \mathbb{R}^{2n+1} \times \{0, 1\}.
\]
(3.2.2) Remarks. (1) The first part of Theorem 6 is probably true also for \( n = 2 \) but we have not written a complete proof for this so far.\(^5\) The second part of Theorem 6 is not true for \( n = 1 \): \( G(S^1) = \text{pt.} \) is 1-colorable but \( S^1 \) knots in \( \mathbb{R}^3 \). However note that any two embeddings of \( S^1 \) in \( \mathbb{R}^3 \) are still isotopic though of course not necessarily ambient isotopic.

(2) If \( X^n = |K^n| \), then \( G(X^n) \) can be identified to a subgraph of \( G_n(K^n) \) by choosing an \( n \)-simplex in each vertex \( X^n \) of \( G(X^n) \). Obviously \( G(X^n) \) is usually much smaller than \( G_n(K^n) \).

(3) And thus Theorem 6 easily checks the embeddability in \( \mathbb{R}^{2n} \) and the unknotting in \( \mathbb{R}^{2n+1} \) of many interesting \( n \)-dimensional spaces. However the sufficient conditions for embeddability and unknotting given in Theorem 6 are far from being necessary. There exist polyhedra \( X^n \), with chromatic number of \( G(X^n) \) arbitrarily big, which embd in \( \mathbb{R}^{2n} \) (or unknot in \( \mathbb{R}^{2n+1} \)). If \( X^n \) is the disjoint union of \( N \) \( n \)-spheres then \( X^n \)-embeds in \( \mathbb{R}^{n+1} \) and \( G(X^n) \) has chromatic number \( N \). Again take the Zeeman Dunce Hat \( Z^2 \). It is a contractible 2-dimensional polyhedron without an edge. If \( X^n \) is the disjoint union of \( N \) copies of \( Z^2 \times S^{n-2} \), \( G(X^n) \) is once again the complete graph on \( N \) vertices. That \( X^n \) unknots in \( \mathbb{R}^{2n+1} \) follows from Price's Theorem [17]: If \( n \geq 2 \) and \( H^n(X^n; \mathbb{Z}) = 0 \), then \( X^n \) unknots in \( \mathbb{R}^{2n+1} \).

(3.3) Equivariant cohomology

We recall some simple facts regarding the cohomology of a polyhedron \( E \) equipped with a fixed point free involution \( \nu \).

(3.3.1) The (singular, integral) cochain complex \( C(E) \) of \( E \), the total space of the 2-fold cover \( \pi: E \to E/\mathbb{Z}_2 \), has two important subcomplexes. The first, \( C_c(E) \), consisting of symmetric cochains \( c \) invariant under the involution of \( E \), \( \nu^*c = c \), can be identified with the pull back of cochains of \( E/\mathbb{Z}_2 \). The second, \( C_a(E) \), consisting of antisymmetric cochains \( c \) which change sign under the involution of \( E \), \( \nu^*c = -c \), can be identified with the pull back of cochains of \( E/\mathbb{Z}_2 \) with twisted integer coefficients \( \hat{\mathbb{Z}} = E \times_{\mathbb{Z}_2} \mathbb{Z} \). The cohomologies of these two subcomplexes can thus be denoted by

\[
H^i(E/\mathbb{Z}_2; \mathbb{Z}) \quad \text{and} \quad H^i(E/\mathbb{Z}_2; \hat{\mathbb{Z}}).
\]

**Lemma 4.** In \( H^n(E^n/\mathbb{Z}_2; \mathbb{Z}) \) or \( H^n(E^n/\mathbb{Z}_2; \hat{\mathbb{Z}}) \), multiplication by \( 2 \) is surjective if it is bijective iff this group is finite and has no elements of order two. Further, this happens iff \( H^n(E^n/\mathbb{Z}_2; \mathbb{Z}) = 0 \).

**Proof.** The first part follows because \( H^n(E^n/\mathbb{Z}_2; \mathbb{Z}) \) and \( H^n(E^n/\mathbb{Z}_2; \hat{\mathbb{Z}}) \) are finitely generated Abelian groups. To see the second part one recalls that

---

\(^5\) A proof of this delicate case is proposed in [34].
the short exact coefficient sequences

\[ 0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}_2 \to 0 \quad \text{and} \quad 0 \to \hat{\mathbb{Z}} \xrightarrow{\times 2} \hat{\mathbb{Z}} \to \mathbb{Z}_2 \to 0 \]

give rise to long exact Bockstein sequences

\[ \ldots \to H^m(E^m/\mathbb{Z}_2; \mathbb{Z}) \xrightarrow{\times 2} H^m(E^m/\mathbb{Z}_2; \mathbb{Z}) \to H^m(E^m/\mathbb{Z}_2; \mathbb{Z}_2) \to 0 \]

and

\[ \ldots \to H^m(E^m/\mathbb{Z}_2; \hat{\mathbb{Z}}) \xrightarrow{\times 2} H^m(E^m/\mathbb{Z}_2; \hat{\mathbb{Z}}) \to H^m(E^m/\mathbb{Z}_2; \mathbb{Z}_2) \to 0. \]

(Note that for \( H^m(E^m/\mathbb{Z}_2; \mathbb{Z}) \) one has an analogue for primes \( p \neq 2 \) also.)

(3.3.2) There is a natural bijective correspondence between equivariant maps \( E \to S^i \) and sections of the \( i \)-sphere bundle \( E \times_{\mathbb{Z}_2} S^i \to E/\mathbb{Z}_2 \) associated to the 2-fold cover \( \pi: E \to E/\mathbb{Z}_2 \). For each \( i \geq 0 \), \( i \) even (resp. \( i \) odd), obstruction theory (Steenrod [23], pp. 177–198, or Milnor [15], pp. 139–148) provides us with an obstruction class

\[ o_{i+1}(E) \in H^{i+1}(E/\mathbb{Z}_2; \hat{\mathbb{Z}}) \quad (\text{resp. } o_{i+1}(E) \in H^{i+1}(E/\mathbb{Z}_2; \mathbb{Z})) \]

which is zero iff this \( i \)-sphere bundle has a cross-section over the \( i + 1 \)-skeleton of any triangulation of \( E/\mathbb{Z}_2 \). In particular if \( m = \dim E \) there is an equivariant map \( E^m \to S^m \). Further, there is a Hopf classification of such maps: the equivariant homotopy classes of equivariant maps \( E^m \to S^m \) can be put in bijective correspondence with the elements of the cohomology group \( H^m(E^m/\mathbb{Z}_2; \hat{\mathbb{Z}}) \) or \( H^m(E^m/\mathbb{Z}_2; \mathbb{Z}) \) depending on the parity of \( m \).

**Lemma 5.** A non-zero obstruction class has order 2.

**Proof.** Under the map \( \pi: E \to E/\mathbb{Z}_2 \), \( o_{i+1}(E) \) pulls back to the \( i + 1 \)-th obstruction class of the pulled back \( i \)-sphere bundle \( \pi^*(E \times_{\mathbb{Z}_2} S^i) \). But this is just the trivial \( i \)-sphere bundle \( E \times S^i \to E \). Thus if \( i \) is even (resp. odd), and antisymmetric (resp. symmetric) cocycle \( \varepsilon \) represents \( o_{i+1}(E) \), then we must have a cochain \( c \) of \( E \) such that \( \delta c = \varepsilon \); therefore \( \delta(c - \nu^*c) = 2\varepsilon \) (resp. \( \delta(c + \nu^*c) = 2\varepsilon \)) and so \( 2o_{i+1}(E) = 0 \).

(3.3.3) One checks easily that cochain complex sequences

\[ 0 \to C_1(E) \subseteq C(E) \xrightarrow{1d - \nu} C_a(E) \to 0 \]
and

\[ 0 \to C_\delta(E) \subseteq C(E) \xrightarrow{1d+\ast} C_\gamma(E) \to 0 \]

are exact. Their long exact sequences, which run

\[
\begin{align*}
(1) \quad & \cdots \to H^i(E/Z_2; Z) \to H^i(E; Z) \to H^i(E/Z_2; \hat{Z}) \\
& \to \cdots, \\
(2) \quad & \cdots \to H^i(E/Z_2; \hat{Z}) \to H^i(E; Z) \to H^i(E/Z_2; Z) \\
& \to \cdots,
\end{align*}
\]

are called the Smith-Richardson sequences (e.g., see Wu [32], Ch. II) of \( E \). One has the following characterisation of the obstruction classes, in terms of the connecting homomorphisms of these sequences:

\[
\begin{align*}
(3) \quad & 0_1(E) = \delta_0^i(1), 0_2(E) = \delta_i^i(0_1(E)), 0_3(E) = \delta_2^i(0_2(E)), \ldots,
\end{align*}
\]

**Lemma 6.**

\[ H^m(E^m/Z_2; Z) \xrightarrow{\ast} H^m(E^m; Z) \quad \text{(resp. } H^m(E^m/Z_2; \hat{Z}) \xrightarrow{\ast} H^m(E^m; \hat{Z})\text{)} \]

is surjective iff it is bijective iff

\[ H^m(E^m/Z_2; \hat{Z}) = 0 \quad \text{(resp. } H^m(E^m/Z_2; Z) = 0\text{)}.
\]

**Proof.** That surjectivity is equivalent to the vanishing of the stated group is clear from the exact sequence (1) (resp. (2)). But this in turn ensures, by Lemma 4, that in \( H^m(E^m/Z_2; Z) \) (resp. \( H^m(E^m/Z_2; \hat{Z})\)) multiplication by 2 is bijective. At the cochain level multiplication by 2 can be written as the composite

\[ C_\delta(E) \subseteq C(E) \xrightarrow{1d+\ast} C_\gamma(E) \quad \text{(resp. } C_\delta(E) \subseteq C(E) \xrightarrow{1d-\ast} C_\gamma(E)\text{)}.
\]

So \( \ast \), which is induced by the first factor \( C_\delta(E) \subseteq C(E) \) (resp. \( C_\delta(E) \subseteq C(E)\)) must be injective.

(3.4) **Kneser deleted product.**

(3.4.1) Before returning to the Kneser graph \( G(X^m) \) let us record some consequences of Weber's Theorem.
Van Kampen-Wu-Shapiro Theorem. If \( n \neq 2 \), \( X^n \) embeds in \( \mathbb{R}^{2n} \) iff the obstruction class \( o_{2n}(X^n) \in H^{2n}(X^n_*/\mathbb{Z}_2; \mathbb{Z}) \) vanishes.

Since there is a \( \mathbb{Z}_2 \)-map \( X^n_* \to S^{2n-1} \) iff \( o_{2n}(X^n_*) = 0 \) this older result is, for \( n \geq 3 \), merely a special case of Weber’s Theorem. If \( n = 1 \) the above characterization of planar graphs follows easily from the well-known Kuratowski Theorem; see Wu [32], p. 210. The 4-dimensional methods of Freedman [5] may have a bearing on the unknown case \( n = 2 \).

Theorem 7. (a) If \( n \geq 2 \), isotopy classes of embeddings \( X^n \to \mathbb{R}^{2n+1} \) are in bijective correspondence with the elements of \( H^{2n}(X^n_*/\mathbb{Z}_2; \hat{\mathbb{Z}}) \).

(b) For \( n \geq 2 \), \( \pi^n \): \( H^{2n}(X^n_*/\mathbb{Z}_2; \mathbb{Z}) \to H^{2n}(X^n_*/\hat{\mathbb{Z}}) \) is surjective iff it is bijective iff \( H^{2n}(X^n_*/\mathbb{Z}_2; \hat{\mathbb{Z}}) = 0 \) iff \( X^n \) unknots in \( \mathbb{R}^{2n+1} \).

(c) For \( n \neq 2 \), \( X^n \) unknots in \( \mathbb{R}^{2n+1} \) only if it embeds in \( \mathbb{R}^{2n} \).

Proof. To see (a) and (b) use Weber’s Theorem in conjunction with (3.3.2) and Lemma 6. (Note that (a) and (b) are not true for \( n = 1 \): a circle \( S^1 \) has \( H^2(S^1_*/\mathbb{Z}_2; \hat{\mathbb{Z}}) = 0 \) but does knot in \( \mathbb{R}^2 \).) To see (c) for \( n \geq 3 \) note that if

\[
H^{2n}(X^n_*/\mathbb{Z}_2; \hat{\mathbb{Z}}) = 0
\]

then, by Lemma 4, \( H^{2n}(X^n_*/\mathbb{Z}_2; \mathbb{Z}) \) is finite with no elements of order two, and so, by Lemma 5, \( o_{2n}(X^n_*) = 0 \). (It would be interesting to have a purely geometric proof of this.) For \( n = 1 \), (c) follows because an \( X^1 \) unknots in \( \mathbb{R}^3 \) iff it has no loops, in which case it does embed in \( \mathbb{R}^3 \).

Price’s Theorem. If \( H^n(X^n; \mathbb{Z}) = 0 \), then \( H^{2n}(X^n_*/\mathbb{Z}_2; \hat{\mathbb{Z}}) = 0 \).

Proof. In fact, \( H^n(X^n; \mathbb{Z}) = 0 \) implies \( H^{2n}(X^n \times X^n; \mathbb{Z}) = 0 \) (because if \( n \)-cochain \( b^n \) is coboundary of \( c^{n-1} \) in \( K^n \), then \( \delta(c^{n-1} \times c^n) = b^n \times e^n \) in \( K^n \times K^n \)) which in turn implies \( H^{2n}(X^n_*/\mathbb{Z}_2; \mathbb{Z}) = 0 \) (by exact cohomology sequence of pair \( (X^n \times X^n, X^n_*/\mathbb{Z}_2) \)) and this implies \( H^{2n}(X^n_*/\mathbb{Z}_2; \hat{\mathbb{Z}}) = 0 = H^{2n}(X^n_*/\mathbb{Z}_2; \mathbb{Z}) \) by using exact sequences (1) and (2) of (3.3.3).

(3.4.2) We define \( X^n_*, \) the Kneser deleted product of the polyhedron \( X^n \), to be the \( \mathbb{Z}_2 \)-subpolyhedron of \( X^n_* \) given by

\[
\bigcup \left\{ X^n_1 \times X^n_j : \{ X^n_1, X^n_j \} \text{ an edge of } G(X^n) \right\}.
\]

\(^6\)A direct proof is proposed in [34].
It is important to observe that in general $X^{n}_{\mathbb{R}}$ does not have the same $\mathbb{Z}_2$-homotopy type as $X^n_{\mathbb{R}}$; e.g., in the case of a circle $S^1$ the deleted product $S^1_\mathbb{R}$ has the homotopy type of $S^1$ while the Kneser deleted product $S^1_{\mathbb{R}}$ is empty. Again, note that the 2-fold cover $X^n_\mathbb{R} \to X^n_{\mathbb{R}}/\mathbb{Z}_2$ is hardly ever trivial; the only connected $X^n_{\mathbb{R}}$'s for which it is trivial are the point and the closed interval. On the other hand the bichromaticity of $G(X^n)$ ensures that the 2-fold cover $X^n_{\mathbb{R}} \to X^n_{\mathbb{R}}/\mathbb{Z}_2$ is trivial.

However we will now check that $X^n_{\mathbb{R}}$ does retain some of the information contained in $X^n_{\mathbb{R}}$.

**Proposition 2.** The inclusions $X^n_{\mathbb{R}} \subseteq X^n_{\mathbb{R}}$ and $X^n_{\mathbb{R}}/\mathbb{Z}_2 \subseteq X^n_{\mathbb{R}}/\mathbb{Z}_2$ induce isomorphisms for the 2n-th homologies and cohomologies.

**Proof.** Choose a triangulation $K^n$ of $X^n$. Since each $X^n_i$ is covered by a subcomplex $K^n_i$ of $K^n$, it follows that $X^n_{\mathbb{R}}$ is covered by a subcomplex $K^n_{\mathbb{R}}$ of $K^n$. Proposition 1 shows that our result will follow if we can show that $K^n_{\mathbb{R}} \subseteq K^n_{\mathbb{R}}$ induces isomorphisms for the 2n-th cohomologies and homologies. Any 2n-cell $\Psi^{2n}_a = a^n \times \theta^n$ of $K^n_{\mathbb{R}} - K^n_{\mathbb{R}}$ either (1) lies in an $X^n_i \times X^n_j$ with $X^n_i \cap X^n_j \neq \emptyset$ or (2) lies in an $X^n_i \times X^n_i$ with $X^n_i \cap X^n_i = \emptyset$ with at least one of the $X^n_i, X^n_j$ (say $X^n_i$) having some points on the edge of $X^n$. Note that int $X^n_i$ (resp. int $X^n_i \times X^n_j$) is a connected open n-dimensional (resp. 2n-dimensional) manifold. In case (1) choose a sequence of (open) n-simplices $a^n = a^n_0, a^n_1, \ldots, a^n_s$ of int $X^n_i$, (resp. $\theta^n = \theta^n_0, \theta^n_1, \ldots, \theta^n_s$ of int $X^n_j$), each sharing a common $(n - 1)$-simplex of int $X^n_i$ (resp. int $X^n_j$) with the next one, such that $a^n_0 \cap \theta^n_0 = \emptyset$ but $a^n_a \cap \theta^n_b = \emptyset$ if $a < r$ or $b < s$. Then

$$\Psi^{2n}_a = \Psi^{2n}_0, \Psi^{2n}_1 = a^n_0 \times \theta^n, \Psi^{2n}_2 = a^n_0 \times \theta^n_1, \Psi^{2n}_3 = a^n_0 \times \theta^n_2, \ldots, \Psi^{2n}_s = a^n_0 \times \theta^n_{s-1}$$

is a sequence of open 2n-cells of $K^n_{\mathbb{R}} \cap \text{int}(X^n_i \times X^n_j)$, each sharing a common $(2n - 1)$-face $\Phi^{2n-1}_{t}, 1 \leq t \leq s - 1$ of $\text{int}(X^n_i \times X^n_j)$ with the next one, with the very last one $\Psi^{2n}_s$ having a $(2n - 1)$-face $\Phi^{2n-1}_s = a^n_0 \times (\theta^n_{s-1} \cap \theta^n_s)$ which is incident to no other 2n-cell of $K$. In case (2) choose a sequence of n-simplices $a^n = a^n_0, a^n_1, \ldots, a^n_s$ of int $X^n_i$, each sharing a common $(n - 1)$-simplex of int $X^n_i$ with the next one, and with the last, $a^n_s$, having an $(n - 1)$-face $\xi^{n-1}_l$ incident to no other n-simplex of $K^n$. Now

$$\Psi^{2n}_a = \Psi^{2n}_0, \Psi^{2n}_1 = a^n_0 \times \theta^n, \ldots, \Psi^{2n}_s = a^n_0 \times \theta^n, \Phi^{2n-1}_s = \xi^{n-1}_l \times \theta^n,$$

has exactly the same properties as before. With appropriate orientations of the cells one has the coboundary formula $\delta(\sum_{s-1} \Phi^{2n-1}_a) = \Psi^{2n}_a$; this shows that any 2n-cochain of $K^n_{\mathbb{R}}$ is cohomologous to one which is supported on $K^n_{\mathbb{R}}$. 


Likewise the boundary formula,

\[
\partial \left( \sum_{a=0}^{l} c_a \Psi^{2n}_a + \text{terms involving other } 2n\text{-cells of } K^n_{\ast} \right)
= \sum_{a=1}^{l-1} (c_a - c_{a-1}) \Phi^{2n-1}_a + c_0 \Phi^{2n-1}_0 + \text{terms involving other (2n-1)-cells of } K^n_{\ast},
\]

shows that \( \Psi^{2n} \) cannot occur (with a non-zero coefficient) in any 2n-cycle of \( K^n_{\ast} \), and thus any 2n-cycle of \( K^n_{\ast} \) is supported on \( K^n_{\ast} \).

With reference to the above proof let us observe that with the appropriate orientations all 2n-cells of each \( X^n_i \times X^n_i \subseteq X^n_{\ast} \) occur with the same coefficient in a 2n-cycle of \( K^n_{\ast} \). Thus each 2n-cycle \( e \in H_{2n}(X^n_{\ast}/\mathbb{Z}_2; R) \) is supported (i.e., has non-zero coefficients) on a \( \mathbb{Z}_2 \)-subspace \( E \) of \( X^n_{\ast} \) which is a union of some \( X^n_i \times X^n_i \subseteq \{ X^n_i, X^n_i \} \) an edge of \( G(X^n) \). For any such \( E \) the corresponding subgraph of \( G(X^n) \) determined by these edges and their vertices will be denoted by \( G(E) \).

**Lemma 7.** \( G(E) \) is bichromatic only if there is a \( \mathbb{Z}_2 \)-map \( f: E \to S^0 \); also conversely provided \( G(E) \) is known to be a full triangle-free subgraph of \( G(X^n) \).

**Proof.** Let us denote \( S^0 \) by \((1,2)\) and let 1 and 2 also denote the two colors. If \( G(E) \) is 2-colored then the \( \mathbb{Z}_2 \)-function \( f: E \to S^0 \) imaging each \( X^n_i \times X^n_i \) to the color of \( X^n_i \) is easily checked to be continuous. Conversely, \( G(E) \) is triangle-free, so \( X^n_i \cap X^n_i = \emptyset = X^n_i \cap X^n_i \) only if \( X^n_i \cap X^n_i \neq \emptyset \), and thus for any \( \mathbb{Z}_2 \)-map \( f: E \to S^0 \) one has \( f(X^n_i \times X^n_i) = f(X^n_i \times X^n_i) \). So we can 2-color \( G(E) \) by assigning to \( X^n_i \) the color \( f(X^n_i \times X^n_i) \).

If each element of \( H_{2n}(X^n_{\ast}/\mathbb{Z}_2; R) \) can be written as a sum of elements \( e \) with chromatic number of \( G(E) \leq k \), then we may say that \( H_{2n}(X^n_{\ast}/\mathbb{Z}_2; R) \) has chromatic number \( \leq k \).

For all \( k \) sufficiently large, \( H_{2n}(X^n_{\ast}/\mathbb{Z}_2; \mathbb{Z}_{2k}) \) is independent of \( k \). This homology group will be denoted by \( H_{2n}(X^n_{\ast}/\mathbb{Z}_2; \mathbb{Z}_{2k}) \).

We now strengthen the first part of Theorem 6 to the following.

**Theorem 8.** If \( H_{2n}(X^n_{\ast}/\mathbb{Z}_2; \mathbb{Z}_{2k}) \) is bichromatic then there is a \( \mathbb{Z}_2 \)-map \( X^n_{\ast} \to S^{2n-1} \). Thus if we further have \( n \geq 2 \) then \( X^n_{\ast} \) embeds in \( \mathbb{R}^{2n} \).

**Proof.** Let us choose \( k \) so large that \( H^{2n}(X^n_{\ast}/\mathbb{Z}_2; \mathbb{Z}) \) has no elements of order \( 2^{k+1} \). The short exact coefficient sequence

\[
0 \to \mathbb{Z}_2 \to \mathbb{Z} \to \mathbb{Z}_{2k} \to 0
\]
induces the Bockstein sequence

\[ \ldots \to H^{2n}(\mathbb{Z}_2; \mathbb{Z}) \xrightarrow{\kappa} H^{2n}(\mathbb{Z}_2; \mathbb{Z}) \to 0. \]

Observe that \( \kappa(\varphi_{2n}(X^n_*)) = 0 \) implies \( \varphi_{2n}(X^n_*) = 2^k \varphi_{2n} \) for some \( \varphi_{2n} \in H^{2n}(\mathbb{Z}_2; \mathbb{Z}) \); but, by Lemma 5, \( 2 \varphi_{2n}(X^n_*) = 2^{k+1} \varphi_{2n} = 0 \), and so, by our choice of \( k, 2^k \varphi_{2n} \), i.e., \( \varphi_{2n}(X^n_*) \), is 0. Thus it would suffice to check that the given hypotheses ensure \( \kappa(\varphi_{2n}(X^n_*)) = 0 \). But \( H^{2n}(\mathbb{Z}_2; \mathbb{Z}) \) is dual to \( H_{2n}(\mathbb{Z}_2; \mathbb{Z}) \) (e.g., see Maunder [14], p. 166). Hence it would suffice to prove that \( \kappa(\varphi_{2n}(X^n_*))(e) = 0 \) whenever \( e \in H_{2n}(\mathbb{Z}_2; \mathbb{Z}) \) has a bichromatic \( G(E) \). Observe that \( e \) lies in the subgroup \( H_{2n}(E; \mathbb{Z}_2) \) and that the restriction of \( \varphi_{2n}(X^n_*) \) to \( E \) is \( \varphi_{2n}(E) \) (this follows for example by (3) of (3.3.3)). Thus,

\[ \kappa(\varphi_{2n}(X^n_*))(e) = \kappa(\varphi_{2n}(E))(e) \]

and it would suffice to prove that \( \varphi_{2n}(E) = 0 \), i.e., that there is a \( \mathbb{Z}_2 \)-map \( E \to S^{2n-1} \). But this follows at once from Lemma 7: \( G(E) \) is bichromatic so there is in fact a \( \mathbb{Z}_2 \)-map \( E \to S^0 \).

(3.5) The deleted join.

(3.5.1) Since the deleted product \( K^n_* \) of a simplicial complex \( K^n \) is only a cell complex, for some purposes it is more convenient to use instead the deleted join \( K^n_* \) of \( K^n \). Take a disjoint copy \( K \) of \( K \). The join \( K \cdot K \) -- i.e., the simplicial complex generated by all simplicies of type \( \sigma \theta \), \( \sigma \in K, \theta \in K \) -- is equipped with the involution \( \sigma \theta \to \theta \sigma \). \( K_* \) is the \( \mathbb{Z}_2 \)-subcomplex of \( K \cdot K \) obtained by omitting those \( \sigma \theta \) for which \( \sigma \cap \theta \) is non-empty. Likewise the deleted join \( X^n_* \) of a space, \( X^* \), is obtained by deleting all points of the type \( \frac{1}{2} x + \frac{1}{2} \bar{x} \) from the join of spaces \( X \cdot \bar{X} \). Analogously to Proposition 1 one can verify that if \( K \) triangulates \( X \), then \( |K_*| \) and \( X_* \) have the same \( \mathbb{Z}_2 \)-homotopy type.

As in [19] we denote by \( \delta_i(K) \) the \( i \)-th least valence (i.e., the least number of \( (i+1) \)-simplices incident to an \( i \)-simplex) of simplicial complex \( K \).

As an illustration of the use of the deleted join functor we prove the following polynomial inequality.

**Theorem 9.** For any \( n \)-dimensional simplicial complex \( K \),

(4) \( \delta_{n-1}(K) < 3(n + 1) \) or \( \left( \delta_{n-1}(K) + n - 1 \right) \leq \dim H_{2n}(K_*; \mathbb{Z}_2) \).

**Proof.** The space \( |K^n_*| \) consists of points of the type \( t x + (1 - t) \bar{y} \), \( 0 \leq t \leq 1 \), where \( x \) and \( y \) lie in disjoint closed simplices of \( |K^n| \). We note
that the subspaces $K_{2,1/2}^n$ and $K_{1,1/2}^n$ of $[K_0^n]$, defined by $t \leq 1/2$ and $t \geq 1/2$ respectively, retracts to the two ends $t = 0$ and $t = 1$, which are homeomorphic to $[K^n]$; and their intersection, defined by $t = 1/2$, is homeomorphic to the deleted product $[K_0^n]$. Thus the Mayer-Vietoris sequence of the pair $\{ K_{2,1/2}^n, K_{1,1/2}^n \}$ yields the exact sequence

\[ \cdots \to H_i(K^n) \oplus H_i(K^n) \to H_i(K^n) \to H_{i-1}(K^n) \to \cdots \]

In particular $H_{n+1}(K^n; \mathbb{Z}_2) \cong H_{2n}(K^n; \mathbb{Z}_2)$; so by applying Theorem 1 of [19] to the simplicial complex $K^n$ we get

\[ \delta_{2n}(K^n) < 2n + 2 \quad \text{or} \quad \delta_{2n}(K^n) + 2n \leq \dim H_{2n}(K^n; \mathbb{Z}_2). \]

The required inequality (4) follows from (6) and the fact (compare (2.2.3) of [20]) that for any simplicial complex $K$,

\[ \delta_{n-1}(K) \leq n \quad \text{or} \quad \delta_{2n}(K^n) \leq \delta_{n-1}(K) \leq \delta_{2n}(K^n) + n + 1. \]

By Theorem 7(a), $K^n, n \geq 2$, unknots in $\mathbb{R}^{2n+1}$ iff $H^{2n}(K^n; \mathbb{Z}_2; \hat{\mathbb{Z}}) = 0$, and so only if $H^{2n}(K^n; \mathbb{Z}_2; \mathbb{Z}_2) = 0$ and so (using exact sequence (1) of (3.3.3) with $\mathbb{Z}_2$ coefficients) only if $H^{2n}(K^n; \mathbb{Z}_2) = 0$. Thus Theorem 9 implies the following result of Sarkaria [20].

**Corollary 1.** If $K^n$ unknots in $\mathbb{R}^{2n+1}$, then $\delta_{n-1}(K) < 3(n + 1)$.

As in [19] the weak $i$-th chromatic number of a simplicial complex, $c_i(K)$, is the least number of colors which can be assigned to the $i$-simplices of $K$ in such a way that no $(i + 1)$-simplex has all its $i$-faces of the same color.

We have conjectured [19] that there exists a constant $C_n$ depending only on $n$ such that $c_{n-1}(K^n) \leq C_n$ for all simplicial complexes $K^n$ embeddable in $\mathbb{R}^{2n}$. In this context we have the following results.

**Corollary 2.** For $n \neq 2$, the class of complexes $K^n$ which unknot in $\mathbb{R}^{2n+1}$ is contained in the class of complexes which embed in $\mathbb{R}^{2n}$ and for this smaller class one has $c_{n-1}(K^n) \leq 3(n + 1)$.

This follows from Corollary 1 and Theorem 7(c).

**Corollary 3.** For $n \neq 2$, the class of complexes $K^n$ for which the Kneser graph of the underlying polyhedron is bichromatic is contained in the class of
complexes which embed in \( \mathbb{R}^{2n} \) and for this smaller class one has \( c_{n-1}(K^n) \leq 6(n+1) \).

In fact if \( G(X_\mathcal{K}^n) \) can be well colored by \( N \) colors, then for each color \( t \), one gets a subpolyhedron \( X_t \subset X_\mathcal{K}^n \) which is the union of all \( X_t^s \)’s colored \( t \). Since any two \( X_t^s \)’s in \( X_t \) intersect Theorem 6 implies that \( X_t \) unknots in \( \mathbb{R}^{2n+1} \) and so the subcomplex \( K^n_\mathcal{K}^n \) of \( K^n \) covering \( X_t \) can have its \((n-1)\)-simplices well colored by \( 3(n+1) \) colors. Taking \( N \) disjoint sets of \( 3(n+1) \) colors to color each \( K^n_\mathcal{K}^n \subset K^n \) we see that the \((n-1)\)-simplices of \( K^n \) can be well colored by \( 3N(n+1) \) colors.

Further results regarding the aforementioned conjecture will be given in a sequel to this paper.  

(3.5.2) Remarks. The following interesting observations regarding the deleted join are due to Flores [4].

1. The deleted join of the \( n \)-skeleton of a \((2n+2)\)-simplex is \( \mathcal{Z}_2 \)-homeomorphic to \( S^{2n+1} \).

By exploiting the formula \((K \cdot L)_n = K_n \cdot L_n \) Flores in fact goes on to give some more examples—the join of \( n + 1 \) copies of three points, the join of \( \sigma_{2n+2}^0 \) and \( \sigma_{2n-k-1}^0 \), etc. of \( n \)-complexes whose deleted join is also \( \mathcal{Z}_2 \)-homeomorphic to \( S^{2n+1} \). (See also Grünbaum [6], exercise 26, p. 67, pp. 210–212, and [7].)

2. The deleted join of a simplicial complex is \( \mathcal{Z}_2 \)-homeomorphic to the deleted product of its cone.

To see this homeomorphism

\[ |K_n| \xrightarrow{\delta} |(v \cdot K)_n|, \]

map each line segment \([x, y]\) to the broken line \([x, v] \cup (x, y)\) \cup \((x, y) \cup (v, y)\) with \( x \) going to \((x, v)\), \((x + y)\) to \((x, y)\), and \( y \) to \((v, y)\). Since \( K^n \) embeds in \( \mathbb{R}^n \) if its cone embeds in \( \mathbb{R}^{n+1} \) it follows from (2) that a \( K^n \) embeds in \( \mathbb{R}^n \) only if there is a \( \mathcal{Z}_2 \)-map from the deleted join of \( K^n \) to \( S^{n} \). So Borsuk’s Theorem implies that the examples of (1) do not embed in \( \mathbb{R}^2 \). Grünbaum [7] proves that if one knocks out an \( n \)-simplex from any of these complexes then the resulting complex embeds rectilinearly in \( \mathbb{R}^n \). Optimal rectilinear immersions (with just one double point) of some of these examples \( K^n \) in \( \mathbb{R}^{2n} \) had been considered also by van Kampen [25] who used them to show \( \delta_{2n}(K^n_\mathcal{K}^n) \neq 0 \) by a direct computation.

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7By incorporating Kalai’s “algebraic shifting” into the above cohomological setup we have now proved the conjectures of [19] as well as (2.5.4)(c); see [34]. However the special cases considered here continue to retain their interest.
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References


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