SOME SIMPLICIAL (CO)HOMOLOGIES

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by

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Introduction. A ubiquitous player in many arguments is the humble SIMPLICIAL COMPLEX, i.e. a (usually finite) set K of finite (nonempty) sets which is closed under inclusion. This homely object is used most often to define and compute some (CO)HOMOLOGY which measures the discrete ambiguity of the (continuous) phenomenon being studied.

My object in these talks is simply to bring out the combinatorial beauty of this methodology. This I'll do by giving many off-beat, but nevertheless elegant and natural, definitions of simplicial (co)homologies, and results concerning their computation. Some of this material - e.g. the cyclic cohomologies of §§ 1 and 7, the philosophy of §§ 3-4, the cyclotomic homology of § 6, and the zeta function
computation of § 8 - is new. More details regarding these results will be given in the lecture notes of my seminar of 1994-95.

I remark that while giving these examples I'll mostly pass over their "standard variations" (dualizing, relativisation, localization, changing coefficients, etc.) in silence, and it will be understood, unless otherwise stated, that the coefficients are always taken from a field $\mathbb{F}$ of characteristic zero.

§ 1. Partial skewsymmetry. Let me begin by recalling how the usual (= oriented) cohomology $H^*(K)$ of a simplicial complex $K$ is defined. We consider the set $K_{assoc}$ of all associative monomials (= finite sequences) in the vertices which are supported on simplices of $K$, and define $C^*(K)$ to consist of all functions $f : K_{assoc} \rightarrow \mathbb{F}$ which are totally skewsymmetric, i.e. such that

$$f(\nu_0 \nu_1 \ldots \nu_q) = (-1)^\pi f(\nu_0 \nu_1 \ldots \nu_q)$$

for all $\pi \in S_{q+1}$. We then define $\delta : C^*(K) \rightarrow C^{*+1}(K)$ by

$$(\delta f)(\nu_0 \nu_1 \ldots \nu_{q+1}) = \sum_i (-1)^i f(\nu_0 \ldots \hat{\nu}_i \ldots \nu_{q+1}),$$

check $\delta \circ \delta = 0$, and set $H^*(K) = \ker \delta / \text{im} \delta$.

If we now drop the above skewsymmetry condition altogether, i.e.
consider the (much bigger !) graded vector space $C^\ast(K_{\text{assoc}})$ of all functions $f : K_{\text{assoc}} \to \mathcal{F}$, and equip it with $\delta$ defined just as above, then this extended map $\delta : C^\ast(K_{\text{assoc}}) \to C^{\ast+1}(K_{\text{assoc}})$ still clearly satisfies $\delta \ast \delta = 0$. We'll denote its cohomology by $H^\ast(K_{\text{assoc}})$.

What is $H^\ast(K_{\text{assoc}})$? More generally, we can ask when partial skewsymmetry, i.e. with respect to permutations $\pi$ belonging to some subgroups $G_{q+1} \leq S_{q+1}$, yields a cochain subcomplex of $C^\ast(K_{\text{assoc}})$, and as to what these cohomologies are? An answer is given by the following.

**Theorem 1.** Partial skewsymmetry with respect to the sequence $(G_{q+1})$ of permutation groups yields a sub cochain complex of $C^\ast(K_{\text{assoc}})$ for all $K$ if and only if $(G_{q+1}) = \{\text{id}\}$ or $(\mathbb{Z}/2)$ (= reversals) or $(C_{q+1})$ (= rotations) or $(D_{q+1})$ (= reversals and rotations) or $(S_{q+1})$. The cohomologies of these five cochain complexes — $C^\ast(K_{\text{assoc}})$, $C^\ast_{\text{rev}}(K_{\text{assoc}})$, $C^\ast_{\text{cyc}}(K_{\text{assoc}})$, $C^\ast_{\text{dih}}(K_{\text{assoc}})$ and $C^\ast_{\text{alt}}(K_{\text{assoc}})$ — are:

$$H^\ast(K_{\text{assoc}}) \cong H^\ast_{\text{rev}}(K_{\text{assoc}}) \cong H^\ast_{\text{alt}}(K_{\text{assoc}}) \cong H^\ast(K),$$

$$H^\ast_{\text{cyc}}(K_{\text{assoc}}) \cong \oplus_{j \geq 0} H^{\ast-2j}(K),$$

$$H^\ast_{\text{dih}}(K_{\text{assoc}}) \cong \oplus_{j \geq 0} H^{\ast-4j}(K).$$

Regarding the proof — for more details read all results of this talk see [15] — note that $C^\ast_{\text{alt}}(K_{\text{assoc}}) = C^\ast(K)$ by definition, while $H^\ast(K_{\text{assoc}}) \cong H^\ast(K)$ is a result of Eilenberg-Steenrod [2]. Using this it
follows easily that the spectral sequence of the \((B,b)\) double complex degenerates at the first term, which gives the cyclic cohomology \(H^*_{\text{cyc}}(K_{\text{assoc}})\). The computation of \(H^*_{\text{dih}}(K_{\text{assoc}})\) is similar.

Thus the generalized De Rham theorem of [16], \(\S\ 5\), is equivalent to saying that above combinatorially defined cohomologies coincide with the corresponding "De Rham" cohomologies defined there.

There are variants of the above game — cf. \(\S\ 7\) — which apply when there is some additional structure. For example if the vertices of \(K\) are colored, then demanding rotational skewsymmetry only with respect to vertices of the same color will give a sub cochain complex.

\(\S\ 2.\) Partial normalization. Many homologies are defined by using a total order on the set of vertices of \(K\). For example using this the set \(K_{\text{comm}}^{*}\) of commutative monomials supported on simplices of \(K\) can be identified with the subset of \(K_{\text{assoc}}^{*}\) consisting of all increasing vertex sequences. Now \(C_{*}(K_{\text{ass}}^{*})\), the vector space spanned by \(K_{\text{ass}}^{*}\), comes equipped with the boundary \(\partial\) dual of \(\delta\). This is given by

\[
\partial(v_{0}v_{1}\cdots v_{q}) = \sum_{i \geq 0} (-1)^{i} v_{0}\cdots \hat{v}_{i}\cdots v_{q},
\]

and so clearly \(\partial\) maps the subspace \(C_{*}(K_{\text{comm}}^{*})\) into itself.

As you know, the homology \(H_{*}(K_{\text{comm}}^{*})\) of \((C_{*}(K_{\text{comm}}^{*}),\partial)\) is
isomorphic to the usual homology $H_*(K)$. However the game is just begun!
There are numerous other natural sub chain complexes of $(C_*(K_{\text{assoc}}), \partial)$
to consider. For instance, for each $r \leq 1$, we have the sub chain complex $(C_*(K_{\text{comm},r}), \partial)$ spanned by all commutative monomials in which
each vertex occurs $\leq r$ times. What are these homologies $H_*(K_{\text{comm},r})$?
The answer is given by the following striking result of T. Bier [1].

**Theorem 2.** For $r$ odd $H_*(K_{\text{comm},r}) \cong H_*(K)$ but for $r$ even

$$H_*(K_{\text{comm},r}) \cong \bigoplus_{\sigma \in K} H_{*-r}\big|_{(\text{Lk}_K})$$

Here the tildes signal "reduced" homologies, i.e. that the empty
set $\emptyset$ is being considered as a simplex. So for $r$ even one of the above
summands is $H_*(K)$ because $\text{Lk}_K \emptyset = K$. Regarding these links $\text{Lk}_K$ we
recall that they are defined as follows. We identify $K$ with the
corresponding element of $C_*(K_{\text{comm},1})$ and divide out completely by $\sigma$ to
get $K = \sigma Q + R$ (so $\sigma$ does not divide $R$). Then $Q = \text{Lk}_K\sigma$.

Curiously it is only the $r$ odd case of Th. 2 which requires some
work. Then the $r$ even case follows easily — again see [15] for details
— by using the fact that, while calculating $\partial$, we can pull out even
powers of vertices (like we pull out constants while differentiating).

**Remark.** The total order of the vertices made $K_{\text{comm}}$ into a
complete semi-simplicial complex with normalization (≠ non-degenerate
simplices) $K_{\text{comm},1}$, so one can call the above $K_{\text{comm},r}$'s partial
normalizations of $K_{\text{comm}}$. Note that these are semi-simplicial complexes,
but in general the non-degenerate simplices of a c.s.s.c. (e.g. those of
assoc) need not constitute a semi-simplicial complex.

§ 3. Deleted joins. Many homologies occur in the following obvious way: one performs a natural combinatorial operation on \( K \) and then calculates homology. This calculation can range from "easy" to "hard" to "very hard".

For instance the join \( K \cdot L \) of two disjoint simplicial complexes (now \( \emptyset \) is a simplex) consists of the unions \((\alpha, \theta)\) of all ordered pairs \( \alpha \in K, \theta \in L \). It is easy enough to compute its homology. One gets for \( K \cdot K \), the join of two disjoint copies of \( K \), the formula

\[
\tilde{H}_* (K \cdot K) \cong \oplus_{p+q+1 = *} \tilde{H}_p (K) \otimes \tilde{H}_q (K).
\]

On the other hand there is no nice formula known for the homology of the (maximal) deleted join \( K \ast K \), i.e. the subcomplex of \( K \cdot K \) consisting of all \((\alpha, \theta)\) with \( \alpha \cap \emptyset = \emptyset \). In fact as I showed in [13] even the case \( \dim K = 1 \) is non-trivial and very interesting.

Remark. \( K \ast K \) is the maximal subcomplex of \( K \cdot K \) on which the \( \mathbb{Z}/2 \)-action \( (\alpha, \theta) \mapsto (\theta, \varphi) \) is free. This is what makes \( K \ast K \) very useful — see e.g. [11] — [13] — for numerous questions re embeddings and colorings of \( K \). We note also that \( K \ast K \) has the \( \mathbb{Z}/2 \)-homotopy type of the space \(|K \cdot K|\) minus its diagonal, and that a nice formula (due to
Richardson-Smith and involving homology operations) is known for the $\mathbb{Z}/2$-homology with mod 2 coefficients of $|K,k|$ mod its diagonal.

We turn now to a smaller deleted join, viz. the subcomplex $K\otimes$ of $K\otimes$ consisting of all $(\sigma,\theta)$ with $\sigma \cup \theta \in K$, for which I got the following formula (it was also noted independently by A. Björner).

**Theorem 3.** $\tilde{H}_*(K\otimes) \cong \bigoplus_{\sigma \in K} \tilde{H}_{*-|\sigma|}(Lk_k\sigma)$.

This is of course remarkably like Bier's theorem except that there $r$ was even and here $r = 1$ (however the combinatorial proof we have is not nice enough to explain this similarity). In § 5 below we'll see that above also follows easily by using Goresky-Macpherson [4].

Recall next that $K$ is called **Cohen-Macaulay** iff the local homologies $\tilde{H}_*(Lk_k\sigma)$ vanish in dimensions less than $\dim K - |\sigma|$. This concept, which is of great importance in combinatorics, has thus the following interesting reformulation in terms of the above homology.

**Corollary 4.** A simplicial complex is Cohen-Macaulay if and only if $\tilde{H}_*(K\otimes)$ vanishes in dimensions less than $\dim K$.

**Remark.** In the above we only considered the case $G = \mathbb{Z}/2$ but one can analogously define various "deleted $G$-joins", $G$ being now any discrete (or even continuous) group, and these too are useful for embedding/coloring questions. We note also that the infinite join $EG =$
§ 4. Assorted visualizations. To understand \( K \circ K \) better (and to define some more "simplicial" homologies!) we now move briefly to the other side of the double-arrow \( \text{DISCRETE} \leftrightarrow \text{CONTINUOUS} \).

We begin by recalling how the usual realization \( |K| \) is defined. One thinks of the \( N \) vertices as the standard bases vectors of \( \mathbb{R}^N \), each subset \( \sigma \) of these is replaced by its convex hull \( \text{Conv}(\sigma) \), and one sets

\[
|K| = \text{Conv}(K) = \bigcup_{\sigma \in K} \text{Conv}(\sigma).
\]

The point which I want to make now is that there is really nothing so sacred about convex hulls!! It is in my opinion at least equally natural to also consider the following affine, linear, spherical, and projective realizations of \( K \).

\[
\text{Aff}(K) = \bigcup_{\sigma \in K} \text{Aff}(\sigma), \quad \text{Lin}(K) = \bigcup_{\sigma \in K} \text{Lin}(\sigma),
\]

\[
\text{Sph}(K) = \bigcup_{\sigma \in K} \text{Sph}(\sigma), \quad \text{Proj}(K) = \bigcup_{\sigma \in K} \text{Proj}(\sigma).
\]

Here of course \( \text{Aff}(\sigma) \) denotes the affine hull of \( \sigma \) in \( \mathbb{R}^N \),
likewise $\text{Lin}(\sigma)$ is its linear hull (= coordinate subspace), $\text{Sph}(\sigma)$ its spherical hull (= unit sphere of $\text{Lin}(\sigma)$), and $\text{Proj}(\sigma)$ its projective hull (= $\text{Lin}(\sigma)$ divided out by the scalar multiplication of $\mathbb{R}^X$).

We note that $\text{Aff}(K)$ has the same homotopy type as $\text{Conv}(K)$ and that $\text{Lin}(K)$ is contractible. However the latter has an interesting singularity at the origin with link homeomorphic to $\text{Sph}(K)$. Note also that $\text{Sph}(K)$ contains a dense subspace homeomorphic to $\text{Aff}(K)$, likewise the projective realization $\text{Proj}(K)$ is also a compactification of $\text{Aff}(K)$.

**Proposition 5.** $K\circ K$ is a $\mathbb{Z}/2$-triangulation of $\text{Sph}(K)$.

So the homology of $K\circ K$ coincides with the singular homology of $\text{Sph}(K)$, and the equivariant homology of $K\circ K$ gives that of $\text{Proj}(K)$. Further we see that $K$ is Cohen-Macaulay iff the singularity of $\text{Lin}(K)$ at the origin is homologically trivial in dimensions less than $\dim K$.

The above proposition follows easily by noting that $K\circ K$ is a union of the octahedral spheres $\bar{\sigma}\circ \bar{\sigma}$, $\sigma \in K$, and we can use these to triangulate the spheres $\text{Sph}(\sigma)$ of $\text{Sph}(K)$.

We note next that there are analogous visualizations $\text{Aff}_F(K)$, $\text{Lin}_F(K)$, etc. over other fields $F$ (for $\text{Conv}_F K$ and $\text{Sph}_F K$ only we need to assume $F$ ordered and normed respectively).

For example, $\text{Sph}_C(K) \subset C^N$ is a union of $(2\dim K + 1)$-dimensional
spheres $\text{Sph}_C(\nu)$, with $S^1$ acting freely on each of these odd spheres via complex multiplication, so the quotient $\text{Sph}_C(K) \to \text{Proj}_C(K)$ is sort of like a "generalized Hopf fibration".

A by-product of all of the above is that we have now a host of new "simplicial" homologies to consider, viz. these visualizations' singular (or even l-adic !) homologies. Many of these, e.g. $H_* (\text{Sph}_K)$, can be computed most conveniently by using the topological methods of either Goresky-MacPherson [4] or Ziegler-Zivaljevic [19] (rather than an explicit triangulation like that of $\text{Sph}_K$ given above).

The most interesting of these computations — see §§ 7 and 8 — is undoubtedly that of $H_* (\text{Proj}_K)$. We'll give in § 7 a small "cyclic" set which triangulates the free $S^1$-space $\text{Sph}_C K$, thus re-interpreting $H_* (\text{Proj}_K)$ as a purely combinatorially defined "cyclic" homology.

Also in § 10 we'll relate the above to the local cohomology of the algebraic variety $\text{Lin}(K)$ at the origin. This cohomology in fact pertains to the commutative algebra $\mathscr{A}(K)$ of all polynomial functions on $\text{Lin}(K)$. Note that $\mathscr{A}(K) = \{ C_*(K_{\text{comm}}) \}$, equipped with the obvious multiplication of commutative monomials).

It would be interesting to compute the Hochschild and cyclic homologies of $\mathscr{A}(K)$, as well as that of the associative algebra $\mathscr{A}_{\text{assoc}}(K)$, obtained by equipping $C_*(K_{\text{assoc}})$ with the multiplication of associative monomials. Since this remains to be done, I will now give
instead a known and very striking non-commutative example.

§ 5. Non-abelian chains. In this section the coefficients are from $\mathbb{Z}$. So $\text{CK} = C_*(K_{\text{comm}})$ is the free Abelian group generated by $K_{\text{comm}}$. We now consider a definition due to Moore which makes sense even for the free (non-Abelian) group $FK = F_*(K_{\text{comm}})$ generated by $K$.

For this we note that the boundary of $C_*(K_{\text{comm}})$ equals

$$\partial = \partial_0 - \partial_1 + \partial_2 - \ldots,$$

where the face homomorphisms $\partial_i : C_*(K_{\text{comm}}) \rightarrow C_{*-1}(K_{\text{comm}})$ are given by

$$v_0 v_1 \ldots v_q \mapsto v_0 \ldots \hat{v}_i \ldots v_q.$$ So $\partial$ maps the subgroup

$$C^\text{Moore}_*(K_{\text{comm}}) = (\ker \partial_1) \cap (\ker \partial_2) \cap \ldots$$

into itself, and on it coincides with $\partial_0$. The key observation of Moore was that the inclusion of chain complexes $(C^\text{Moore}_*(K_{\text{comm}}), \partial_0) \rightarrow (C_*(K_{\text{comm}}), \partial)$ induces an isomorphism in homology $H^\text{Moore}_*(\text{CK}) \cong H_*(K)$.

This is very interesting because (unlike the bigger chain complex) the non-Abelian analogue of this smaller chain complex, i.e.

$$(F^\text{Moore}_*(K_{\text{comm}}) = (\ker \partial_1) \cap \ker \partial_2 \cap \ldots, \partial_0),$$

is still a chain complex, whose homology will be denoted $H^\text{Moore}_*$.
That $\partial_0$ is a differential of $F_*^{\text{Moore}}(K^{\text{comm}})$ is clear from $\partial_0 \partial_0 = \partial_0 \partial_1$, and to check that its image is a normal subgroup one needs a small verification which uses the usual relations between the $\partial_i$'s and the degeneracy homomorphisms $s_j : F_*(K^{\text{comm}}) \to F_{*+1}(K^{\text{comm}})$, $v_0 \cdots v_q \mapsto v_0 \cdots v_j \cdots v_q$.

Thus we can in fact define Moore homology $H_*^{\text{Moore}}(G_*)$ for any simplicial group $G_*$, i.e. a sequence $G_n$, $n \geq 0$, of groups equipped with face and degeneracy homomorphisms satisfying the usual relations.

With the definition made, there arose the usual question: what is $H_*^{\text{Moore}}(FK)$? The striking answer was soon discovered by Milnor [10].

Theorem 6. $H_*^{\text{Moore}}(FK) \cong \pi_{*+1}(S^0.K)$, i.e. the Moore homology gives the homotopy groups of the suspension of $K$.

We note that the above — see Kan [6] for more — gives a purely combinatorial definition of these homotopy groups which is much more satisfying than the obvious one (e.g. by using a huge "Kan extension" of $K_{\text{assoc}}$) suggested by the simplicial approximation theorem. Also it tends to show that, despite appearances, the homotopy groups are indeed the natural "non-Abelian analogues" of the homology groups!

§ 6. Higher order boundaries. Let $p \geq 2$ and let us assume that the characteristic zero coefficients $F$ contain the $p$th roots of unity.
We now equip $C_*(K_{assoc})$ with the cyclotomic boundary defined by

$$\partial(v_0 v_1 \ldots) = \sum \omega^r (v_0 v_1 \ldots \hat{v}_r \ldots),$$

where $\omega = \exp(2\pi i/p)$, $p \geq 2$. This boundary also obviously preserves the subspace $C_*(K_{comm})$ spanned by monomials which are increasing with respect to a given total order of the vertices. Moreover an easy calculation, using $(1 + \omega + \omega^2 + \ldots + \omega^{p-1}) = 0$ shows $\partial^p = 0$.

So for each ordered pair $(r,s)$ of positive integers with sum $r + s = p$ we can define the cyclotomic homology groups

$$H_{*;r,s}(K_{comm}) = \frac{\ker(\partial^r)}{\text{im}(\partial^s)}.$$

**Theorem 7.** The non-zero cyclotomic homology is as follows:

$$H_{kp+r-1; r, s}(K_{comm}) \cong H_{kp+s-1; s, r}(K_{comm}) \cong H_{2k}(K),$$

$$H_{kp-1; r, s}(K_{comm}) \cong H_{kp-1; s, r}(K_{comm}) \cong H_{2k-1}(K).$$

For the proof see [15]. Over cyclotomic integers this cohomology is more involved and not yet fully worked out.

I found the above definition — actually a more general one using any non-trivial character $\omega$ of any finite group — while going over the first Complément of Poincaré's *Analysis Situs* during my seminar [14] of 1993–94. Thus this definition was inspired by the very first
Remark. If we use mod $p$ coefficients, and replace $-1$ in the usual definition of $\partial$ by (the trivial character) $1$, then too we have $\partial^p = 0$. This Mayer homology [8] was first computed by Spanier [17]. The results are similar to those given above. It seems that the game of § 1 can also be generalized to group characters other than $-1$.

§ 7. Cyclic deleted joins. We want to combinatorialize the free $S^1$-space $\text{Sph}_C(K)$. For $K = \text{pt}$, $\text{Sph}_C(K) \cong S^1$, and we'll use the discrete circle of Connes, i.e. the completion $\mathcal{S}$ of the semi-simplicial complex having one vertex $\emptyset$ and one edge $(\emptyset\emptyset)$, together with a rotational group structure $\cong \mathbb{Z}/r+1$ on the $r+1$ $r$-simplices of $\mathcal{S}$ for each $r \geq 0$. The definition of these \textit{"rotations"} $t$ should be clear from the following — here $\emptyset(\emptyset\emptyset) = \frac{1}{2}\emptyset\emptyset(\emptyset\emptyset)$ etc. — which shows the case $r = 3$.

$$\alpha = (\emptyset\emptyset)\emptyset\emptyset, \ t(\alpha) = \emptyset(\emptyset\emptyset)\emptyset, \ t^2(\alpha) = \emptyset\emptyset(\emptyset\emptyset), \ t^3(\alpha) = \emptyset\emptyset\emptyset\emptyset.$$ 

For a general $K$ we use a union of some joins of disjoint copies (each augmented by $\emptyset$) of $\mathcal{S}$ to define

$$K \otimes C = \bigcup_{\emptyset \in K} \left( \prod_{\emptyset \in \alpha} \mathcal{S}_\alpha \right).$$

Now let $\delta : C^*(K \otimes C) \rightarrow C^{*+1}(K \otimes C)$ be the sum of the maps induced by the coboundaries $\delta_\emptyset : C^*(\mathcal{S}) \rightarrow C^{*+1}(\mathcal{S})$. One has $\delta^2 = 0$ and $\delta$ preserves the subspace $C^*_\text{cycl}(K \otimes C)$ of cochains $f$ rotationally
skew-symmetric over each $S_\nu$, i.e. such that $f(\sigma)$ only gets multiplied by $(-1)^{\dim \nu}$ when the part $\sigma_\nu$ of $\sigma$ is "rotated" to $t(\sigma_\nu)$. We'll denote the cohomology of $(C^*_\text{pcycl}(K\!\text{-}\!\mathbb{C}K), \delta)$ by $H^*_{\text{pcycl}}(K\!\text{-}\!\mathbb{C}K)$.

**Theorem 8.** The partially cyclic cohomology $H^*_{\text{pcycl}}(K\!\text{-}\!\mathbb{C}K)$ gives the singular cohomology of the projective realization as follows:

$$H^*_{\text{pcycl}}(K\!\text{-}\!\mathbb{C}K) \cong \bigoplus_{j \geq 0} H^{*-2j}_{\text{proj}}(\text{Proj}_{\!\mathbb{C}K}).$$

One also has $H^*(K\!\text{-}\!\mathbb{C}K) \cong H^*(\text{Sph}_{\!\mathbb{C}K})$, in fact the Milnor realization $|K\!\text{-}\!\mathbb{C}K|$ of $K\!\text{-}\!\mathbb{C}K$ is a cell subdivision of $\text{Sph}_{\!\mathbb{C}K}$ which "complexifies" the triangulation $K\!\text{-}\!K$ of $\text{SphK}$ as follows.

We put the vertex $0_\nu$ of $S_\nu$ on the identity $1_\nu \in \mathbb{C}_\nu$ of the $\nu$th coordinate axis of $\mathbb{C}^N$, and lay the edge $(00)_\nu$ of $S_\nu$ over the unit circle $S^1_\nu$ of $\mathbb{C}_\nu$. Then clearly the non-degenerate simplices of $K\!\text{-}\!\mathbb{C}K$, i.e. those with all parts equal to $0_\nu$ or $(00)_\nu$, furnish us with a cell subdivision of $\text{Sph}_{\!\mathbb{C}K}$. The intersection of this cell subdivision with $\mathbb{R}^N$ gives a triangulation $\cong K\!\text{-}\!K$ of $\text{SphK}$.

The proof of Theorem 8 is analogous to how cyclic cohomology is interpreted as an equivariant $S^1$ cohomology (see e.g. Loday [7]). We use a bigger (but homotopically equivalent) realization $\|K\!\text{-}\!\mathbb{C}K\|$. This turns out to be $\text{Sph}_{\!\mathbb{C}K} \times ES^1$ and on it the combinatorial rotations...
translate into the diagonal \( S^1 \)-action. So the required cohomology interprets as the singular homology of the quotient \( \text{Proj}_C K \times BS^1 \).

The combinatorial chain complex of Theorem 8 can probably be used to give a new proof of the following known result of Ziegler-Zivaljevic (and possibly Goresky-MacPherson and others).

**Theorem 9.** \( H_*(\text{Proj}_C K) \cong \bigoplus_{j \geq 0} H_{*-2j}(K^{(j)}) \).

Here \( K^{(j)} \) is the \( j \)th co-skeleton of \( K \), i.e. the poset (under \( \leq \)) of all simplices of \( K \) having dimensions \( \geq j \), and \( H_*(K^{(j)}) \) denotes the homology of the order complex of this poset.

In fact the order complex is a key example of a Grothendieck realization \( \|..\| \). It is the realization \( \|K^{(j)}\| \) of the category obtained from \( K^{(j)} \) by thinking of each \( \leq \) as a morphism. (There are also \( \|..\| \)'s of functors, etc. According to Professor Burghelea this concept is essentially due to Ehresmann. Many people have contributed to the theory of these realizations, notably Segal and Bousfield-Kan.)

Since the Ziegler-Zivaljevic [19] proof of Theorem 9 also makes a heavy use of these realizations, a couple of quick words re them seem to be in order. Grothendieck's main idea was simply to use as simplices some trains of morphisms, with faces defined in the obvious way by "erasing" one object at a time. So e.g. the train
has as faces the four trains shown below.

Thus "Hochschild faces" arise naturally if we visualize categories (or functors, etc.) à la Grothendieck et al.

§ 8. Zeta functions. In this section I'll use the finite field of \( q \) elements \( \mathbb{F}_q \) (and its finite extensions \( \mathbb{F}_{q^n} \)). I'll give a result which has a well known "explanation" involving the (definition of the) \( \ell \)-adic cohomology of \( \text{Proj}_F K \). Since this cohomology is isomorphic to \( H^*_{\mathbb{Q}}(\text{Proj}_C K) \), it is natural to wonder whether this "explanation" can also be combinatorialized à la § 7?

Riemann's zeta function \( \zeta(s) = \prod \frac{1}{1 - p^{-s}} \) generalizes from \( \mathbb{Z} \) to any ring \( R \) with finite quotient fields \( R/M \) by replacing the \( p \)'s by the cardinalities \( |R/M| \). For example \( R \) can be the coordinate ring \( \mathcal{M} \) of any affine variety (e.g. \( \text{Lin}_F K \) over \( \mathbb{F}_q \); now each \( |R/M| = q^n \) for some \( n \), and \( M \) determines and is determined by a point of the variety over the finite extension \( \mathbb{F}_{q^n} \). So the definition generalizes still further to
all projective varieties (e.g. $\text{Proj}_q \mathbb{P}^1$) over $\mathbb{F}_q$ : use in place of $\mathcal{M}$'s points over the finite extensions $\mathbb{F}_{q^n}$ and in place of $p$'s the cardinalities of the residue fields at these points.

Thanks to the fundamental theorem of arithmetic, Riemann's zeta is also given by $\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$. Likewise, thanks to a unique factorization theorem for ideals, the aforementioned zeta function $\zeta_K(s)$ of $\text{Proj}_q \mathbb{P}^1$ is also given by

$$\zeta_K(s) = \exp \left( \sum_{n \geq 1} a_n(K) \frac{(q^{-s})^n}{n} \right),$$

where $a_n = |\text{Proj}_{\mathbb{F}_{q^n}} \mathbb{P}^1|$. 

**Theorem 10** (with Anders Björner). Let $f_i(K)$ denote the number of $i$-simplices of $K$. Then

$$\zeta_K(s) = \prod_{j \geq 0} \frac{1}{(1 - q^{-s}) e_j(K)},$$

where $e_j(K) = \sum_{i \geq j} (-1)^{i-j} \binom{i}{j} f_i(K)$ = Euler characteristic of the order complex of the $j$th co-skeleton of $K$.

For this we substitute $a_n(K) = \sum_{i \geq 0} f_i(K) (q^n - 1)^i$ in the formula for $\zeta_K(s)$ and simplify. The identification of the numbers $e_j(K)$ with the Euler characteristics $e(K^{(j)})$ requires some more work.
Alternatively, using $Z_K(t)$, the function of $t$ obtained from $\zeta_K(s)$ by replacing $q^{-s}$ by $t$, we can write the above formula as

$$Z_K(t) = \frac{P_1(t) P_2(t) P_3(t) \cdots}{P_0(t) P_1(t) P_2(t) \cdots},$$

where $P_i(t)$ is the following integral polynomial of degree equal to the $i$th Betti number of $\text{Proj}_CK$:

$$P_i(t) = (1-t)^{b_i(K(0))} (1-qt)^{b_i(K(1))} (1-q^2t)^{b_i(K(2))} \cdots.$$ 

Grothendieck's "explanation" of such factorizations runs thus:

The cohomology of the complexified variety can be computed $l$-adically. If $l$ does not divide $q$ then $x \mapsto x^q$ and its iterates act on this $l$-adic complex. Lefschetz's fixed point formula implies factorization.

We note that $\text{Proj}_CK$ is singular — however its singularities have codimension $\geq 2$, i.e. it is a pseudomanifold — and that the functional equation and Riemann hypothesis do not hold for $\zeta_K(s)$. However these probably hold for a suitably modified zeta function based on the homologies considered in the next section.

§ 9. Other skeleta. Let $\pi$ denote a total order (= permutation) of $[n] = \{0, 1, \ldots, n\}$. We define the $i$th $\pi$-skeleton of an $n$-dimensional simplicial complex $K$ by
\[ K^i_\pi = \{ \sigma : \pi^{-1}(\text{dim} \sigma) \leq i \}. \]

For example, \( K^1_{\text{id}} \) is the (usual) 1st skeleton \( K^1 \), while for the opposite total order \( \sigma(\text{id}) = (n, n-1, \ldots, 1, 0) \) one has \( K^1_{\sigma(\text{id})} = K^{(n-1)} \), the \((n-1)\)th co-skeleton of \( K \).

For each \( i \) the inclusion \( K^i_\pi \to K_{\pi}^{i+1} \) induces a simplicial map in the order complexes of these posets. Using this we define the \( \pi \)-homology of \( K \) by

\[ H^\pi_i(K) = \text{im} \left( H_i(K^i_\pi) \to H_i(K_{\pi}^{i+1}) \right). \]

Note that \( H^\text{id}_1(K) \) identifies with the usual homology \( H_1(K) \), which of course depends only on \( 1 \) and the underlying space of \( K \). For \( \pi \neq \text{id} \) only partial answers are known re the topological invariance of these homologies \( H^\pi_1(K) \).

**Theorem 11.** Let \( K \) be an \( n \)-dimensional pseudomanifold and let \( \pi \) or its opposite \( \sigma(\pi) \) be such that it ends with \( (\ldots, n-1, n) \) and has just one local minimum. Then \( H^\pi_1(K) \) depends only on \( \pi, 1, \) and the underlying space of \( K \), and one has \( H^\pi_1(K) \cong H^\sigma_{n-1}(K) \).

The above is due to Goersky-MacPherson [3] who call \( H^\pi_1(K) \)'s of the above kind the intersection homologies of \( K \). (To correlate our definition with theirs we note that each \( \pi \) of the above kind determines

\[ \text{20} \]
a "perversity" p under which our $q^1_{\pi}$ becomes their "basic set" $q^1_{\pi}$.

If $K$ is a $n$-manifold, i.e. if for each $\sigma \in K$ the trains of $\|K\|

having all elements $\geq \sigma$ determine a dual cell of dimension $n-\dim(\sigma)$,
then $H^{n-1}_n(K)$ is isomorphic to the usual cohomology $H^{n-1}_n(K)$. Thus the
above result contains Poincaré's duality theorem $H_i(K) \cong H^{n-1-i}(K)$. In
fact intersection homology was found in the course of trying to "better
understand" the cell/dual-cell proof of this theorem which Poincaré gave
in the first Complément of his Analysis Situs !

§ 10. Algebras. The next example is that of "a cohomology with
local coefficients" (these go back to Čech and Steenrod). Let $\mathcal{A} = \mathcal{A}(K)$
be the algebra of all polynomial functions defined on $\mathrm{Lin}(K)$. For each
simplex $\sigma$ the coefficient ring $\mathcal{A}_\sigma$ will be zero unless $\sigma \in K$ when it will
consist of all rational functions defined on $\mathrm{Lin}(\sigma)$.

Let $C^q(\mathcal{A})$ consist of all skew-symmetric functions $X$ from vertex
sequences with values $X(v_0v_1...v_q)$ in $\mathcal{A}$ where $\sigma = (v_0, v_1, ..., v_q)$. Let $\delta : C^q(\mathcal{A}) \rightarrow C^{q+1}(\mathcal{A})$ be defined by

$$(\delta X)(v_0v_1...v_q) = \sum_i (-1)^i \phi_i(X(v_0...\hat{v}_i...v_q)),
$$

where $\phi_i$ denotes the obvious map from the coefficient ring of the face
of $\sigma$ opposite $v_i$, to the coefficient ring of $\sigma$. One has $\delta \circ \delta = 0$ and $\ker(\delta)$ will be called, following Grothendieck, the local cohomology $H^*(\mathcal{A})$.
of $\text{Lin}(K)$ at the origin (or of $\mathcal{A}$ at its irrelevant ideal).

All nonzero elements of the type $[v_1^\lambda \ldots v_N^\lambda] \in \mathcal{A}_\lambda$, $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{Z}^N$, constitute a vector space basis of $\mathcal{A}_\lambda$. Using this we get a finer $\mathbb{Z}^N$-grading $C^q(\mathcal{A}) = \bigoplus_\lambda C^q_\lambda(\mathcal{A})$ and $H^q(\mathcal{A}) = \bigoplus_\lambda H^q_\lambda(\mathcal{A})$, where summation is over all $\lambda \in \mathbb{Z}^N$ with $|i : \lambda_i \neq 0| = q+1$. The following result is due to Hochster (see Stanley [18]).

**Theorem 12.** $H^q_\lambda(\mathcal{A}_K)$ can be nonzero only if all $\lambda_i \leq 0$ and $\sigma = (v_1^\lambda : \lambda_i \leq -1) \in K$ in which case $H^q_\lambda(\mathcal{A}_K) \cong H^{q-1-|\sigma|}(\text{Lk}_K \sigma)$.

Using §§ 3–4 it follows that this algebraic local cohomology, though infinite dimensional, is determined by the singular local cohomology $H^q(\text{Lin}_K, \text{Lin}_K \setminus \{0\})$ of $\text{Lin}(K)$ at $\{0\}$.

Note that Krull dimension $\dim \mathcal{A} = \dim K + 1$, and that $H^q(\mathcal{A}) = 0$ for all $q > \dim \mathcal{A}$. If $H^q(\mathcal{A}) = 0$ also for all $q < \dim \mathcal{A}$ then one says that $\mathcal{A}$ is a Cohen-Macaulay ring (there are various other equivalent reformulations of this concept). So the above theorem implies the following very useful result of Reisner.

**Corollary 13.** The ring $\mathcal{A}(K)$ is Cohen-Macaulay if and only if the simplicial complex $K$ is Cohen-Macaulay in the sense of § 3.

We note that the defining cochain complex $C^*_*(\mathcal{A})$ of local cohomology is isomorphic to the limit of the cochain complexes $C^*_*(\mathcal{A})_k$, $k$
where $C^q_k(\mathcal{A})$ consists of all skew-symmetric functions $X$ from length $q + 1$ vertex sequences to $\mathcal{A}$, and $\delta : C^*_k(\mathcal{A}) \to C^{*+1}_k(\mathcal{A})$ is defined (somewhat like the "De Rham" $\delta$ of [16], § 5) by

$$(\delta X)(v_0, v_1 \ldots, v_q) = \sum_i (-1)^i (v_i)^k \cdot X(v_0 \ldots \hat{v_i} \ldots v_q).$$

The resulting Koszul cohomologies $H^*_k(\mathcal{A})$, which are finite-dimensional approximations of the local cohomology $H^*(\mathcal{A})$, are also quite interesting combinatorially, e.g. Hochster showed that

$$H^*_1(\mathfrak{A}_K) \cong \oplus_{\sigma} H^{*-1-|\sigma|}(K\setminus\sigma),$$

where now $\sigma$ runs over all subsets of $\{v_1, \ldots, v_N\}$, and $K\setminus\sigma$ denotes the subcomplex of $K$ consisting of all $\Theta$ disjoint from $\sigma$.

There are also associated to $\mathcal{A}$ — and likewise to the algebras of the varieties $\text{Aff}(K)$ and $\text{Proj}(K)$ — many other (co)homologies, e.g. the algebraic De Rham cohomology, Hochschild homology, cyclic homology, etc., and there are some (or many, depending on one's viewpoint) results scattered in the literature regarding their computations.

REFERENCES


[16] ——, From calculus to cyclic cohomology, talk given in this Workshop.


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