# Table of Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abstract</td>
<td>111</td>
</tr>
<tr>
<td>Table of Contents</td>
<td>vi</td>
</tr>
<tr>
<td>The Dukhan Cohomology of Foliated Manifolds</td>
<td>1</td>
</tr>
<tr>
<td>Introduction (A) Motivation</td>
<td>1</td>
</tr>
<tr>
<td>(B) Summary of Results</td>
<td>3</td>
</tr>
<tr>
<td>Text</td>
<td>17</td>
</tr>
<tr>
<td>1. The filtration</td>
<td></td>
</tr>
<tr>
<td>2. The filtered complex</td>
<td>18</td>
</tr>
<tr>
<td>3. The spectral groups $L^k_1$</td>
<td>19</td>
</tr>
<tr>
<td>4. Newb foliation</td>
<td>22</td>
</tr>
<tr>
<td>5. The spectral homomorphisms $d_0$</td>
<td>23</td>
</tr>
<tr>
<td>6. Invariant transverse forms</td>
<td>26</td>
</tr>
<tr>
<td>7. $E^k_{transverse}$</td>
<td>26</td>
</tr>
<tr>
<td>8. Invariant complex line bundles</td>
<td>29</td>
</tr>
<tr>
<td>9. $k$-homology and $k$-chain homology</td>
<td>33</td>
</tr>
<tr>
<td>10. Construction of homotopies</td>
<td>35</td>
</tr>
<tr>
<td>11. The cup product</td>
<td>37</td>
</tr>
<tr>
<td>12. Signature vanishing</td>
<td>39</td>
</tr>
<tr>
<td>13. Generalized Bott vanishing</td>
<td>41</td>
</tr>
<tr>
<td>14. A Bigradation</td>
<td>46</td>
</tr>
<tr>
<td>15. The $E_2$ term</td>
<td>49</td>
</tr>
<tr>
<td>16. Functional analytic preliminaries</td>
<td>53</td>
</tr>
<tr>
<td>17. 2-parametrized and global parallalisms</td>
<td>63</td>
</tr>
<tr>
<td>18. Serr dualty, Kodaira-Bochner complete sets</td>
<td>69</td>
</tr>
<tr>
<td>19. Connection theory</td>
<td>75</td>
</tr>
<tr>
<td>20. Bott connections, strict bundles etc.</td>
<td>76</td>
</tr>
<tr>
<td>21. Wall morphisms</td>
<td>83</td>
</tr>
<tr>
<td>22. Foliations as torsionless Structures</td>
<td>92</td>
</tr>
<tr>
<td>23. 2-Parametrized on Principal Bundles</td>
<td>100</td>
</tr>
<tr>
<td>Bibliography</td>
<td>111</td>
</tr>
</tbody>
</table>
The Deligne Cohomology of Foliated Manifolds

Introduction (1) We first point out in broad outline the object of our study, and why it is worth studying.

Think of a p-vector at \( x \in M \), where \( M \) is a smooth n-dimensional manifold, as an antisymmetric and multilinear map \( T^1_x \cdots T^p_x \rightarrow T^0_x \) (p times) \( w_p : T^1_x \otimes \cdots \otimes T^p_x \rightarrow T^0_x \) being the tangent space at \( x \). We shall say that \( w_p \) is of filtration \( \geq 1 \) with respect to a subspace \( E_p \subset T^0_x \) if it vanishes whenever \( p-1+1 \) of the arguments are in \( E_p \).

Let us now suppose that \( M \) is supplied with a tangent subbundle \( D \subset T \). Then a form \( w \) will be said to be of filtration \( \geq 1 \) if \( w \) is such for each \( x \in M \). Thus we get a decreasing sequence of vector spaces

\[ V = V_0 = V_1 = \cdots = V_{p-1} = F, \quad V_j \text{ being all smooth forms of filtration } \geq 1. \]

This sequence is known as the filtered complex \( D \). The \( p \)-th term is the homology of \( D \). The exterior derivative preserves the filtration.

Our starting point is the simple observation that the exterior derivative preserves the filtration. So each \( Ax \) is now a subcomplex of \( Ax \), and a is a filtered complex. On the other hand, by Frobenius theorem, \( \Delta = \sum H(\cdot) - D \) is involutive if and only if it is tangent to a foliation; in other words, \( M \) can be covered by coordinate neighborhoods \( \ldots, \lambda_1, \lambda_2, \ldots, \lambda_k \), such that locally \( \lambda_1 \lambda_1, \ldots, \lambda_1 \lambda_k \) form a basis for \( E_1 \). In this way \( M \) is partitioned into 1-dimensional manifolds called the leaves of the foliation; each leaf being a maximal connected sub-manifold given locally by some constant values for \( \lambda_1 \lambda_1, \ldots, \lambda_1 \lambda_k \).

Hence to each pair \((M, F)\), (i.e. to a foliated manifold \( M \)) is attached in a natural way a filtered complex \( A \). Our object is to study this filtered complex. We recall that by standard homological algebra, as in [7], one can attach to any filtered complex an object called its spectral sequence.

It consists of a sequence \( E \), of graded groups, each of which is the homology of the preceding under a differential \( d_\ast \), such that \( E_\ast \) is the graded group of \( A \) (under above filtration) and the final term \( E_\infty \) (which is attainable in a finite number of steps) is the graded group of \( H(A) \) (under the obvious induced filtration).

A better idea of the importance, and scope, of such an investigation can be formed by considering the analogous case of complex manifolds. In this case, we have similar spectral sequence with \( E_0 = H(A) \) (i.e., the complex de Rham cohomology, and \( A \) is the so-called...
Néron cohomology. A vast theory is centered around this cohomology (see, e.g., [13]). Deep results requiring both analytical and algebraic techniques have been attained. For instance, one has the finiteness theorem (i.e., if \( \Sigma \) is compact \( \Sigma \) is finite dimensional), the duality theorem (i.e., for \( H^1 \) compact \( H^1 = H^1 \cdot \text{dim}\Sigma \cdot \text{dim}\Sigma \); and on the algebraic side, one can mention results involving characteristic classes (e.g., for \( \Sigma \) compact \( \Sigma \) compact \( \Sigma \) and \( \text{dim}\Sigma \); this is called the Hironaka-Roch theorem). These three specimen results due respectively to Cartan-De Rham, Serre and Hirzebruch have in turn led to very interesting generalizations due to Grothendieck-Weil, Grauert, Grothendieck and Grothendieck-Atiyah-Singer and others.

The results, concepts and constructions occurring in this work should all be viewed as part of an ongoing and extensive program whose goal is to build a similar body of knowledge for foliated manifolds.

(3) The following is a summary of the contents.

Sections 1-5 give a rapid review of the apparatus of our spectral sequence, following Cartan-Eilenberg [7].

In sections 6-7 we show that \( H^1 = H^1 \). Here \( \mathcal{P} \) is the sheaf of germs of smooth forms of degree \( p \) which are transverse and invariant. A p-form \( \omega \) is called transverse if \( \omega \in \mathcal{P} \), it is called invariant if \( \mathcal{P} = 0 \) for any vector field \( X \in \mathcal{O}(\mathcal{P}) \). \( \mathcal{P} \) -- or just \( \mathcal{P} \) -- is simply the sheaf of germs of smooth functions which are constant on leaves. It plays an important role in many places.

Section 8 studies complex line bundles. Let \( \mathcal{O}_\mathcal{P} \) (resp. \( \mathcal{O}_\mathcal{P} \)) denote the sheaf of germs of smooth nonzero complex valued functions (resp. those that are constant on leaves). Now, the isomorphism classes of such bundles form a group under tensor product, viz., \( \mathcal{O}_\mathcal{P} \times \mathcal{O}_\mathcal{P} \); and the sheaf inclusion \( \mathcal{O}_\mathcal{P} \rightarrow \mathcal{O}_\mathcal{P} \) gives us a map \( \mathcal{O}_\mathcal{P} \rightarrow \mathcal{O}_\mathcal{P} \) whose image gives us the invariant line bundle. The first Chern class of a line bundle \( \xi \in \mathcal{P}(\mathcal{O}_\mathcal{P}) \) is an element \( \xi \in H^1(\mathcal{P}) \). We shall say that a Chern class vanishes in \( \mathcal{P} \) if it is killed by the induced map \( H^1(\mathcal{P}) \rightarrow H^1(\mathcal{P}) \). We see that the first Chern class of a line bundle vanishes in \( \mathcal{P} \) if and only if it is invariant. For an analogous result for analytic line bundles over a complex manifold, see [15].

In sections 9, 10 we study two notions of
homotopy. In the category of foliated manifolds the
morphisms are those which map leaves into leaves. Two
such maps \( f, g: \mathcal{M} \rightarrow \mathcal{N} \) are called
\( k \)-homotopic \((k-\text{id})\) if they can be extended to a
morphism \( f \times g: \mathcal{X} \times \mathcal{I} \rightarrow \mathcal{N} \) where
\( \mathcal{X} \times \mathcal{I} \) carries the \( k \)-foliation \((k-\text{id})\). Here the
1-
foliation is given by multiplying each leaf of \( \mathcal{M} \) by \( 1 \), and
the 2-foliation is the natural \( 1 \)-dimensional foliation on \( \mathcal{X} \times \mathcal{I} \). Then we construct a \( k \)-chain
homotopy (between the induced maps \( f_{*}, g_{*} \) of filtered comple
xes) by using a \( k \)-homotopy, i.e., a
chain homotopy that "disturbs" filtration by \( k-1 \)
units. Thus we can think of the spectral sequence \( E_{k} \),
for \( p \geq k \), as a functor attached to the \( k \)-homotopy
category \((k=1,2)\) of foliated manifolds, by usual
homological algebra [(7)].

Sections 11, 12 involve some simple observations about
the relationship of the exterior product to the
filtration. Since our filtration is of length \( \ell \),
if a form of filtration \( i \) is multiplied by a form of
filtration \( j \) and \( i+j = \ell \) we get 0. Using this we see
that an odd-dimensional filtration can have
\[ \dim H^{2k}(\mathcal{M}) = \frac{1}{2} \dim H^{2k}(\mathcal{M}) \text{ only if } \ell = 2 \cdot \text{ even}. \]
Here \( H^{2k}(\mathcal{M}) \) is the part of \( H^{2k}(\mathcal{M}) \) of filtration \( 2k-1 \).

In section 13 we extend the definition of
section 8 to define \textit{Invariant complex vector bundles}.
Let \( \mathcal{D}_{0} \) (resp. \( \mathcal{D}_{0} \)) denote the sheaf of germs of smooth
function with values in \( GL(n, \mathbb{C}) \) (resp. those that are
constant on leaves). Now the isomorphism classes
form only a set \( H^{1}(\mathcal{M}, \mathcal{D}_{0}) \), and we consider the image of
\[ H^{1}(\mathcal{M}, \mathcal{D}_{0}) = H^{1}(\mathcal{M}, \mathcal{D}_{0}) \text{ of all invariant bundles; this}
\text{notion shall play a role in section 19. Further it}
\text{plays a role in a natural } K \text{ theory of invariant}
\text{bundl}es.\] We show that the \textit{real Chow ring of an}
\textit{invariant complex vector bundle vanishes in dimensions}
\( n \geq 3 \). Applying this result to the complexification of
the bundle \( \Omega^{n} \) of transverse 1-forms one gets Bott's
vanishing theorem [4]. Note that no connection theory
is used in the proof.

Corresponding to any two projection maps
\( \mathcal{P}_{2}, \mathcal{P}_{3}: \mathcal{M} \rightarrow \mathcal{N} \) with images \( D \) and \( \mathcal{X} \) s.t. \( D + \mathcal{X} = \mathcal{M} \), one has a
natural \textit{filtration} of \( \mathcal{X} \) and the differential \( \partial \) is
the sum of three differentials \( \delta_{1}, \delta_{2}, \delta_{3} \) of

In section 16 we show that $E_2 = H_{10}(E)$ and $E_2 = H_{10}(M)$. [Following (7), this means that $E_1, E_2$ are the first 2 terms of the spectral sequence of the double complex $(d_{ij}, d_{kl}).$] The complex of sheaves $\mathcal{A}_j \to \mathcal{A}_{j+1} \to \ldots \to \mathcal{A}_n \to 0$ (which is in fact exact) has 3 standard spectral sequences; see, e.g., [15]. Using the notation of (13), the $E_2$ term of the "power" spectral sequence in case $E_2$ of our spectral sequence. This is shown in section 15. We see from these that if the foliation arises from a fibration the $E_2$ term is the same as that in Serre's spectral sequence of a fibration [11].

Section 16 covers the functional analysis that is relevant to finiteness theorems. We use a theorem for the usual Fréchet space topology. So it induces on each $E_2$ a topological vector space topology. We recall the usual definition of a smoothing map (= integral operator) via $F$. Any K-chain homotopy between 1 and $\varepsilon$ will be called a $k$-parametric (of $\varepsilon$). We show, under some additional conditions, that on a compact foliated manifold, the existence of a $k$-parametric implies that $E_2$ is finite dimensional. This result is the reason why we shall be interested in $k$-parametrices. We also point out in this section that if $\dim E_2 = e < \infty$, then $\chi(M) = \pi_1^{-1}(\pi_0(M))$. Here $\chi(M)$ is the Euler characteristic of $M$.

In section 17 we construct, by modifying techniques given in [11], a $k$-parametrices in the following special cases: assume that there exists a contiguously uniformly transitive map $F: \mathbb{R}^n \rightarrow \mathcal{C}^k(M, \mathbb{R})$, with $F(0) = 1$, then there exists a $k$-parametrices. Here $\mathcal{C}^k(M, \mathbb{R})$ consists of all smooth maps $M \rightarrow \mathbb{R}$ which map leaves into leaves; it is given the usual $\mathbb{R}^n$ topology. By uniformly transitive we mean that there is a neighborhood $U$ of $0 \in \mathbb{R}^n$ which, for all $x \in U$, gives us a diffeomorphism $F_x$ of $U$ onto a neighborhood of $x$ by $F_x(-) = F(0)(x)$. We point out that this hypothesis is fulfilled if a compact foliated manifold $M$ has a global foliation by vector fields which are infinitesimal transformations of the foliated structure.

Section 18 records some tidbits which may be of value. $\varepsilon_{1,2}$ we point out that an obvious extension of a "spreading argument" of [11] allows us to show that if the foliation arises from a fibration then there is a $k$-parametrices. Again, now let's suppose that $k$ is oriented. If the differentials $d_{0,1,2,\ldots}$ of the
spectral sequence are assumed to be topological homomorphisms than the foliation obeys Serre's duality. This hypothesis is satisfied if dim $E_i < \omega$. Hence odd-dimensional foliations with dim $E_i < \omega$ exist only if signature $(\omega) = 0$. We shall also recall in this article an example which shows that for almost all irrational flows on the torus $E_i < \omega$. But there exist irrational flows for which this is no longer true.

Section 19 studies the inter-relationships between reduction of the structure sheaf of a bundle--see [12a] and sections 8, 13 above--and connection theory. In section 19a we think of a connection on a real vector bundle $W$ over $N$ as a derivation of $\mathcal{A}(W)$, $\mathcal{A}(W)$ to $\mathcal{A}(W)$, lying above $d$ (p. 75). Here $\mathcal{A}(W)$ are forms on $N$ with coefficients in $W$. Now the "structure sheaf" of $W$ consists of the smooth germs with values in $\mathcal{O}(W)$, $w = \text{dim } W$. Again, as in sections 8, 13, $W$ will be called an invariant vector bundle if this sheaf $\mathcal{O}(W)$ can be reduced to $\mathcal{O}(W)_{\mathbb{G}}$, the sub-sheaf of germs constant on leaves. We put a bigrading in the manner of section 14 now and thus a splits up into three derivations $\delta_0, \delta_1, \delta_2$. Then $W$ is invariant if and only if one has a connection for which $\delta_0 = 0$. Such a connection will be called a Bott-connection. Note that one can now define $\mathcal{H}(W)$, the homology of $\mathcal{A}(W)$ under $\delta_0$. Also if $W$ is the natural $\omega$-dimensional bundle associated to $W$, (p. 32), we can define the group $\mathcal{H}(W)$. For each $\xi \in H^1(W, \mathcal{O}(W))$ we shall also define the notions $\xi$-invariant connection (resp. $\xi$-invariant connection) by requiring that the local connection matrices of $1$-forms (with respect to trivializations of $\xi$ agreeing with $\xi$) consist of transverse (resp. transverse invariant) 1-forms. Any vector bundle $W$ associated to $\xi$ admits a $\xi$-Bott connection, but it need not admit an $\xi$-invariant connection. Note that our definition of Bott connection merely says that the curvature form (which is a 2 form with coefficients in $\omega$) is of filtration $\geq 1$. If we satisfy "filtration $\geq 2$" we shall say that it is an invariant connection. Finally, we call a bundle which admits an $\xi$-invariant connection a stiff bundle. Starting with any Bott connection one can define an element $[\xi_{+1}]$ of $H^1(W)$ by using the part of the curvature which is of bidegree $1,1$. In complete analogy with Atiyah ([1] it will be seen that $W$ is stiff if and only if $[\xi_{+1}] = 0$.

Section 19b is devoted to examining the wall
homomorphisms, and is basically a dual of section 19A. Let G be a Lie group. (For simplicity we think of G as a matrix group.) By a G-algebra we mean a graded anticommutative algebra (over some commutative ring) which is provided with (a) a differential d_i for each left invariant vector field X on G and (b) a skew derivation l_a of degree -1 and (c) a derivation l_a of degree zero such that l^2_a = 0, l_a l_b = [l_a, l_b], l_a 0 = 0, l_a 1 = 0. For example the enveloping algebra W(G) of G is a G-algebra—see [6]. Another example arises from A(P), the Lie algebra of a principal G-bundle P over K. In this case each left invariant vector field X gives us in a natural way a vector field X—along the fibre and l_a X are defined to be the usual inner product and Lie differentiation respectively. We can define a connection to be a G-algebra morphism of W(G) into some other algebra. There always exist such morphisms W(G) \rightarrow A(P); this definition is known to be same as the definition (10) for G = O(2). A connection represents a codimension 2 filtration, so A(P) is a filtered complex. W(G) is also a filtered complex if we set W(n) as all those terms containing polynomials of degree \geq n; this is called the filtration of W(G). When we are considering W(G) with this filtration we'll write (1) A(P). Then we will see that a principal G-bundle is invariant if and only if there is a connection (1) A(P) \rightarrow A(P) preserving the filtrations. Of course "invariant" means, as before, that we can reduce the structure group H to G. (This result is simply the dual of the first in the above paragraph; in this new setting these are the best connections.) For each \xi \in \Omega^1(H, \pi_1) we have a family of 1-forms \xi(x)^1(x) = \Delta(P) which are associated to \xi (They arise in the proof of above proposition). We show that any such connections \xi(x) gives G(P) to be isomorphic (in the sense of section 9). Thus to each \xi \in \Omega^1(H, \pi_1) there is associated a cohomology class [\xi(x)'1(x)] \in A(P) or (5) connections. This implies that, for \xi \neq 1, the induced maps \Delta(P) \rightarrow \Delta(P) depend only on \xi; we can denote it by \Delta. Then dualising a result of section 19A we can see that, an invariant bundle (\xi, \mu(\Delta(\pi_1))) is unique if the map \mu(\xi)'1(x) \rightarrow \Delta(P) is zero. We now define the cohomology of W(G) by putting \mu(\xi)'1(x) = \Delta(\xi)'1(x). Then we can deduce that a principal G-bundle is invariant if and only if we have a connection \xi(x) \rightarrow A(P) preserving the filtration. (These are what were in section 19A the invariant)
connections. Note that a Bott connection is simply one in which the curvature is of filtration $g_1$, while an invariant connection is one in which it is of filtration $g_2$. For each $t \in \mathfrak{h}(\mathbf{G}, \mathfrak{g}_0)$ there is a family of $t$-invariant connections $\tilde{\gamma}(t)$. We show that any $2$ of them are 2-homotopic (section 9). Thus to each $t \in \mathfrak{h}(\mathbf{G}, \mathfrak{g}_0)$, there is associated a 2-homotopy class $\tilde{\gamma}(t)$, which reduces to $\tilde{\gamma}(t)$ of invariant connections. In particular, for $P \in \mathcal{S}(\mathfrak{g})$, the induced map $(\tilde{\gamma}(t), \tilde{\gamma}(P), \tilde{\gamma}(t), \tilde{\gamma}(P))$ depends only on $t$. We can denote it by $\tilde{\gamma}(t)$. Note that for a point foliation, $G_0 = G_2$, and for each differentiable structure $\tilde{\gamma} \in \mathfrak{h}(\mathbf{G}, \mathfrak{g}_0)$, one has the well-known map $(\tilde{\gamma}(t), \tilde{\gamma}(P), \tilde{\gamma}(t), \tilde{\gamma}(P)) = (\tilde{\gamma}(t), \tilde{\gamma}(P), \tilde{\gamma}(t), \tilde{\gamma}(P))$ called the Chern-Weil homomorphism. Now $\tilde{\gamma}(t)$ means $\tilde{\gamma}$ considered with the foliation arising from the fibration $P \to \mathfrak{g}$.

In section 20 we study some aspects of linear connections that are relevant to our study—and find use in section 21—we follow standard terminology, as in [17]. A linear connection is any connection on the principal tangent bundle (and so on its associated vector bundles, $T_1, T_2$ etc.). Given any pair $(\mathbf{N}, \mathfrak{g})$, i.e., a manifold, plane field one says that a linear connection is a Kaluza connection see [17]. [35]—if it is torsionless and reducible to $\mathbf{g}^0$ (i.e., keeps the plane field $\mathbf{g}$ parallel). Then a Kaluza connection exists if and only if $\mathbf{g}$ is involutive. With section 19 in mind it is natural to say that a linear connection which reducible to $\mathbf{g}^0$ (an invariant bundle) is a Bott connection if it restricts to such a connection on $\mathbf{g}^0$.

Every Kaluza connection is a Bott connection. This section points out that foliated manifolds may be studied in the context of torsionless $\mathfrak{g}$-structures; however I have not pursued this aspect in this work.

Section 21 is occupied with constructing $\mathfrak{g}$-parametrices (see section 17 above) for foliated principal bundles. Special hypotheses are needed for these constructions. E.g., the hypothesis made in section 21a is that—in the terminology of section 20—the foliation arises from a torsionless $\mathfrak{g}(1,\mathfrak{g})$-structure. Here $\mathfrak{g}(1,\mathfrak{g})$ is the Lie algebra of all endomorphisms of $\mathbf{g}$ with image in $\mathbf{g}^0$. Let $\mathfrak{q}$ denote the principal bundle of compatible frames, provided with the natural codimension $c$ foliation. Then we employ the canonical parallelism of $\mathfrak{q}$ by such a torsionless connection (assumed complete) to construct a $\mathfrak{g}$-parametrix on $\mathfrak{q}$ a la section 17. In section 21b we assume that $\mathbf{g}^0$ is stiff; then we can have a Kaluza connection whose restriction to $\mathbf{g}^0$ is an invariant...
connection (i.e., filtration of curvature $\geq 2$; see section 1) a linear connection with the latter property can be called an invariant connection. Let $P$ denote the principal bundle of tangent frames compatible with the foliate structure. Then we employ the canonical parallelism of $P$ by such an invariant torsionless connection (assumed complete) to construct a parallelism for $E$.

In this result $E$ is not foliated in dimension $c$. Instead we show that there is always a natural 1+1-dimensional foliation of $P$ sitting above the foliation of $M$. This is the foliation that occurs in the above result. In section 3 we use averaging process (over a Haar measure) to get a $\delta$-parametrix for the subalgebra $A_\delta$ of forms which are right invariant over $G$.

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Let $M$ be a smooth compact $m$-dimensional manifold. We shall denote its tangent bundle by $T$. The dual cotangent bundle is $T^*$. By $p$-forms we shall understand smooth sections of $\Lambda^p T^*$, the bundle of $p$-covectors; the space of all $p$-forms is denoted by $\Lambda^p$, and the space of all forms by $\Lambda$. Let us now assume given, once and for all, a subbundle $E = E$ of fibre dimension $1$ and codimension $1$ so $1 + 0 = n$. We now define a filtration of $\Lambda$:

$$A = A_1 = A_2 = \ldots = A_2 = 0$$

(1)

in the following way: we think of a $p$-form $\theta$ as a multi-linear skew-symmetric form on the tangent vector fields, $\omega(x_1, \ldots, x_p)$, which commutes with the action of $C^\infty(M)$, i.e., $\omega(x_1, \ldots, x_k, \ldots, x_p) = \delta_k^i(x_1, \ldots, x_p)$. The values of this form are in $C^\infty(M)$. We now say that $A_p$ consists of all those $p$-forms which vanish at $x \in M$ whenever $p-1+1$ of the vector fields $x_1, x_2, \ldots, x_p$ lie in $E$ at $x$. If $p \geq 0$ it is seen that $A_0 = 0$. We shall understand by $A_1$ the space of forms lying in $A_p$ for some $p$. If $p \leq 0$, $A_1 = \Lambda$ and if $p \geq 0$, $A_1 = 0$. Another way of looking at this filtration is this. We have an isomorphism $\Lambda^p T^* \cong (\Lambda^p T)^*$ at $x \in M$, an elt., in the latter bundle is a linear map $\Lambda^p T^* \to \mathbb{R}$. By putting the requirement that this linear map vanish on $\Lambda^p \wedge \ldots \wedge \Lambda^p$ whenever $p-1+1$ of the vectors are in $E$ we get a subbundle $\Lambda^p E$. The space of all sections of this subbundle is $\Lambda^p E$. We also have the bundle $\Lambda^p T^*$ formed from the Whitney sum $\bigoplus \Lambda^p T^*$. The space of all sections of this is $\Lambda^p$.

2. Now we assume that the subbundle $E$ is involutive, i.e., if two tangent vector fields $X$ and $Y$ take their values in $E$ so does their Lie bracket $[X, Y]$. In the real vector space $\Lambda$ we have the exterior derivative, $d(\omega)$ which is given by the following formula (thinking of forms as skew-symmetric multilinear maps $C^\infty(M)$ x $\ldots$ x $C^\infty(M)$ = $C^\infty(M)$ as above explained):

$$d(\omega)(x_0, x_1, \ldots, x_p) = \frac{1}{p+1} \left( (-1)^k x_i(\omega(x_0, \ldots, \hat{x}_i, \ldots, x_p) \right)$$

(2)

+ $\sum_{k \leq i \leq p} \omega(x_i x_j, x_0, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_p)$

Though the notion of $d$ goes back at least to H. Cartan, the intrinsic definition (2) was given first by Palais [22].

Proposition 1. $E$ is involutive if and only if the filtration (1) commutes with the endomorphism $d$, i.e., $d(\omega) \in A_1$ for all $i$.

Proof: If $E$ is involutive we easily see that if $(p+1)-i+1$ of the vectors $x_0, x_1, \ldots, x_p$ are in $E$ then $p-1+1$ of the vectors $x_0, \ldots, \hat{x}_i, \ldots, x_p$ and
In the following we will denote the homologies induced by $d$. In case other differential $d$ enters the picture we will use the notation $H_d$. Also in the following we will frequently use the triangle lemma (CE, p. 316) which says that if (in the figure shown), the bottom row is exact then $\frac{\ker d}{\ker d_1} \cong \ker d_0$.

$$\begin{array}{c}
\alpha' \to \alpha \\
\downarrow \quad \downarrow \\
\alpha'' \\
\end{array}$$

Since our filtration is compatible with $d$, we have an induced filtration

$$H(\alpha) \supseteq H_1(\alpha) \supseteq \ldots \supseteq H_n(\alpha) = 0$$

of the homology of $A$, where $H_i(\alpha) = \ker (H_{i+1}(\alpha) \to H_i(\alpha))$. The 2 filtrations (1) and (2) give rise to the associated quotients $E_2^{p,q} = \frac{H_p(\alpha)}{H_{p+q}(\alpha)}$.

And furthermore for each $x \geq 1$ we put

$$E_2^{p,q} = \frac{H_p(\alpha)}{H_{p+q}(\alpha)}$$

where the 2 morphisms are the connecting homomorphisms in the exact homology sequences arising from the exact
sequences

\[ 0 \rightarrow A_{11} \rightarrow A_{11} \rightarrow A_{11} \rightarrow 0 \]

and

\[ 0 \rightarrow A_{11} \rightarrow A_{11} \rightarrow A_{11} \rightarrow 0 \quad \text{respectively} \]

The numerator and denominator in (4) shall be denoted \( A^1 \) and \( B^1 \) respectively. One notes that \( A^1 \rightarrow A^2 \rightarrow \cdots \rightarrow A^r \rightarrow 0 \) and the spaces \( K_i \) are increasing with \( i \) while the spaces \( Z_i \) are decreasing with \( i \). If \( r \) is bigger than or equal to either of the two numbers \( s \) and \( s + 1 \), (or briefly, if \( r \) is big) then these \( 2 \) spaces stabilize and (4) reads

\[
\text{Im} \left( \frac{A_{11}}{A_{11}} \right) = \text{Im} \left( \frac{A_{11}}{A_{11}} \right)
\]

(To see this read numerator of (4) as \( \text{Im} \left( \frac{A_{11}}{A_{11}} \right) \), take \( r \) big now). Since \( 0 \rightarrow A^1 \rightarrow A^2 \rightarrow \cdots \rightarrow A^r \rightarrow 0 \) is exact, this coincides with

\[
\text{Im} \left( \frac{A_{11}}{A_{11}} \right) \rightarrow \text{Im} \left( \frac{A_{11}}{A_{11}} \right).
\]

This in turn coincides with

\[
\text{Im} \left( \frac{A_{11}}{A_{11}} \right) = \text{Im} \left( \frac{A_{11}}{A_{11}} \right), \quad \text{i.e.}, \quad A^1.
\]

We see that if \( r \) is big \( E_r = A^1 \). This property is called the convergence of the spectral sequence.

In addition our vector space \( s \) is graded by the degree of forms. And we have an induced filtration

\[ H^n(s) = H^n(s) = \cdots = H^n(s) = 0 \quad (3') \]

for each \( p \), where \( H^n(s) = \text{Im} \left( H^n(s) \rightarrow H^{n+1}(s) \right) \).

Similarly for each \( p \) we have the filtration

\[ x^p = A^p = \cdots = A^1 = 0 \quad (4') \]

Associated to these \( 2 \) sets of filtrations are the quotients \( A_{11}^p = \frac{A_{11}}{A_{11}} \) and \( z_{11}^p = \frac{H^n(s)}{H^{n+1}(s)} \). And

\[ x^p = A^p = \cdots = A^1 = 0 \quad (4') \]

for each \( p \), we have

\[ x_{11}^p = \frac{\text{Im} \left( \frac{A^p}{A_{11}} \right)}{\text{Im} \left( \frac{A^p}{A_{11}} \right)} \quad \text{and} \quad \frac{H^n(s)}{H^{n+1}(s)} \]

\[ x_{11}^p = \frac{\text{Im} \left( \frac{A^p}{A_{11}} \right)}{\text{Im} \left( \frac{A^p}{A_{11}} \right)} \quad (4') \]

Once more, we point out that the vector spaces \( x_{11}^p \)

(b) One known from de Rham's theorem that \( H_i(s) \) is just \( H_i(M, \mathbb{R}) \), the real cohomology of the manifold, which is a topological invariant. We see thus that

\[ x_{11}^p = \bigoplus_{i=0}^{n-1} x_{11}^i \quad (5) \]

The whole idea of the spectral sequence is to find the
Interplay between $E_r$ and $\mathbb{R}_r$ for low $r$. For this purpose, more algebra is introduced. As explained below, in each $E_r$ a differential $d_2$ of degree $r$ is introduced (i.e., $d_2^0 = 0$ and $d_2^1 = d_2^{14}$) and it is shown that $\mathbb{R}_r = E_{r+1}$. Many times this statement itself—quite independent of the nature of the differentials $d_2$—suffices to calculate some of the groups entering into the spectral sequence. For example, let us consider Serre's (23) filtration of $\mathbb{R}^3$, when $\mathbb{R}^{13}$ can be non-zero only if $0 \leq 1$ or $0 \leq 2$. From (5) we see that $\mathbb{R}^{0,0} = \mathbb{R}^{1,2} = \mathbb{R}$ and the other $\mathbb{R}^{1,1} = 0$. Supposing we have shown that $\mathbb{R}^{1,0} = \mathbb{R}$ (section (6) below). Then $d_2^{1,0} = 0$, must be the null map and we must have $\mathbb{R}^{1,0} = 0$.

2. To wind up the algebraic machinery of the spectral sequence, we recall the definition of $d_2$. For this one needs, using the exactness of

$$0 \to \frac{\mathbb{A}_1}{\mathbb{A}_1 + \mathbb{A}_1} \to \frac{\mathbb{A}_1}{\mathbb{A}_1 + \mathbb{A}_1} \to \frac{\mathbb{A}_1}{\mathbb{A}_1 + \mathbb{A}_1} \to 0,$$

that

$$\frac{\mathbb{A}_1}{\mathbb{A}_1 + \mathbb{A}_1} \to \frac{\mathbb{A}_1}{\mathbb{A}_1 + \mathbb{A}_1} \to \frac{\mathbb{A}_1}{\mathbb{A}_1 + \mathbb{A}_1} \to \frac{\mathbb{A}_1}{\mathbb{A}_1 + \mathbb{A}_1} \to 0,$$

and

$$\text{ker}(d_2) \text{ equals } E_{r+1}, \text{ i.e., } E_r \to E_{r+1}.$$
called invariant if $A_{x} = 0$ for any $x \in C^{n}(\Sigma)$. When $
abla$ is involutive, we can (by Frobenius theorem) find local coordinates $x_{1}, \ldots, x_{k}; x_{1}^{1}, \ldots, x_{k}$ such that the leaves are given by assigning some constant values to the last $k$ of these. In terms of these local coordinates an invariant transverse $p$-form will appear as

$$\sum_{1 \leq j_{1} \leq \ldots \leq j_{p} \leq m} f_{j_{1}, \ldots, j_{p}}(x_{1}, \ldots, x_{k}) \, dx_{j_{1}} \wedge \cdots \wedge dx_{j_{p}}$$

where $1 \leq j_{1} \leq \ldots \leq j_{p} \leq m$. One notes that the coefficiens $f_{j_{1}, \ldots, j_{p}}$ are constant on each leaf. When we change coordinates to another compatible system of same type the transformation matrix thus consists of functions which are constant on leaves.

Now we define $\mathcal{F}$ to be the vector space of all invariant transverse $r$-forms. The corresponding sheaf of germs of such forms will be denoted by $\mathcal{F}_{r}$. In particular $\mathcal{F}_{0}$, or just $\mathcal{F}$, is the sheaf of germs of functions which are smooth and constant on leaves. As usual if a leaf $\Sigma$ sits over our manifold $\lambda$, $\mathcal{K}_{r}(\lambda, \Sigma)$ shall denote the sheaf cohomology of $\lambda$ with coefficients in the sheaf $\mathcal{F}_{r}$.

We remark in passing that the transverse invariant forms are those which remain parallel along leaves with respect to any Bott connection on the bundle of transverse forms (see def. of Bott connection in section 19). Also we will notice as we proceed further that functions which are constant on leaves play a part as important for foliated manifolds as that of analytic functions in complex manifolds.

The following proposition gives the first term of the spectral sequence.

**Proposition 2.** $\mathcal{K}_{r}^{0}(\lambda) \cong H^{0}(\lambda, \mathcal{F}_{r})$.

The next section deals with the proof of this proposition. Note that the Reeb foliation of $\mathbb{S}^{3}$ and more generally the foliations of solid given by Thurston [36]. Manas [36] are such that any global smooth function which is constant on leaves is simply a constant. In other words $\mathcal{K}_{r}^{0} = H^{0}(\lambda, \mathcal{F}) = 1$ for such foliations. Note that by 'thickening' the compact leaves one can destroy this property.

**L. Proof of Proposition 2.** We recall that $\mathcal{K}_{r}^{0}$

$$\mathcal{K}_{r}^{0}(\lambda) \overset{\text{def}}{=} \frac{A_{\lambda}}{\frac{\partial}{\partial \lambda}}.$$  

We now construct a sheaf $\mathcal{K}_{r}^{0}$ in the following way. We take the presheaf which assigns to each open set $U$ of $\lambda$ the vector space $\mathcal{K}_{r}^{0}(U)$. If $V \subset U$ we have natural restriction maps $\mathcal{K}_{r}^{0}(U) \rightarrow \mathcal{K}_{r}^{0}(V)$ which yield isomorphisms $\mathcal{K}_{r}^{0}(U) \cong \mathcal{K}_{r}^{0}(V)$. $\mathcal{K}_{r}^{0}$ is taken to be
the sheaf determined by this presheaf. If \( U \) has local coordinates \( x_1, x_2, \ldots, x_n, y \), we may represent the stalk at \( x \) by expressions of the type
\[
\omega_{a_0}(x, y)dx_{a_1} \wedge \cdots \wedge dx_{a_q} \wedge dy_{b_1} \wedge \cdots \wedge dy_{b_p}. \tag{7}
\]
It is understood that for the multi-indices \( a \) and \( b \) we have \( a_1 < \cdots < a_q \) and \( b_1 < \cdots < b_p \). In this local representation the zeroth differential
\[
\partial_0 : \mathfrak{E}^0(U) \rightarrow \mathfrak{E}^1(U)
\]
is given by
\[
\partial_0[\omega_{a_0}(x, y)dx_{a_1} \wedge \cdots \wedge dx_{a_q} \wedge dy_{b_1} \wedge \cdots \wedge dy_{b_p}]
\]
\[
= \frac{1}{(q + p)!} \sum_{k=1}^{n} \sum_{i=1}^{m} \frac{\partial \omega_{a_0}(x, y)}{\partial x_k} \partial x_k \wedge dx_{a_1} \wedge \cdots \wedge dx_{a_q} \wedge dy_{b_1} \wedge \cdots \wedge dy_{b_p}. \tag{8}
\]
And so we also have a parallel sheaf homomorphism
\[
\partial_0 : \mathfrak{E}^0(U) \rightarrow \mathfrak{E}^1(U). \tag{9}
\]
Using this we construct the following sequence
\[
0 \rightarrow \mathfrak{E}^0(U) \rightarrow \mathfrak{E}^1(U) \rightarrow \mathfrak{E}^2(U) \rightarrow \cdots \rightarrow \mathfrak{E}^{m+p}(U) \rightarrow 0. \tag{10}
\]
Note that \( \mathfrak{E}^0(U) \) is simply the sheaf of germs of smooth \( p \)-forms which vanish where one of the vectors is in \( D \).

This explains the first inclusion. We prove now that this sequence (10) is exact. An element of \( \mathfrak{E}^0(U) \) is given locally by sum of terms of the type \( \omega_{a_0}(y)dx_{a_1} \wedge \cdots \wedge dx_{a_q} \wedge dy_{b_1} \wedge \cdots \wedge dy_{b_p} \). It is clear that \( \partial_0 \) will kill it. Conversely an element of \( \mathfrak{E}^0(U) \) is sum of terms of the type
\[
\omega_{a_0}(x, y)dx_{a_1} \wedge \cdots \wedge dx_{a_q} \wedge dy_{b_1} \wedge \cdots \wedge dy_{b_p}
\]
and it is clear from (7) that \( \partial_0 \) will kill such a sum only if each \( \omega_{a_0} \) is a function of \( y \) alone. This shows exactness at first place. Since \( \partial_0 \) is zero in \( U \) by (7), to show exactness at other places, we assume that
\[
\partial_0[\Sigma \omega_{a_0}(x, y)dx_{a_1} \wedge \cdots \wedge dx_{a_q} \wedge dy_{b_1} \wedge \cdots \wedge dy_{b_p}] = 0
\]
which can happen only if
\[
\partial_0[\Sigma \omega_{a_0}(x, y)dx_{a_1} \wedge \cdots \wedge dx_{a_q}] = 0
\]
for each multi-index \( a \). Using Poincaré's lemma for each \( y \) one can find a q-form \( \Xi(x, y) dx_{a_1} \wedge \cdots \wedge dx_{a_q} \) such that
\[
\partial_0[\Sigma \omega_{a_0}(x, y)dx_{a_1} \wedge \cdots \wedge dx_{a_q}] = 0.
\]

The construction of these functions \( \Xi(x, y) \)---where, e.g., \( \text{exteriorproduct}[\cdot \cdot \cdot ] \)---only involves integrating the smooth functions \( \omega_{a_0}(x, y) \) and their derivatives over \( x \). So these functions can be chosen to be smooth in both \( x \) and \( y \). Since we have
by using (9) and (10) it follows that the sheaf sequence (8) is exact. Note that $\mathcal{F}^{D} \otimes \mathcal{G}$ arises also as the sheaf of germs of smooth cross-sections of a vector bundle.

So it is fine, i.e., any local cross-section of $\mathcal{F}^{D} \otimes \mathcal{G}$ can be extended globally. Again for the case smooth, the space of smooth sections of $\mathcal{O}^{D} \otimes \mathcal{G}$ is precisely $\mathcal{O}^{D} \otimes \mathcal{G}$.

These two facts, the fine sheaf resolution (8), and standard sheaf theory—see e.g., Hirsch [12]—now imply that the cohomology $H^*(\mathcal{F}, \mathcal{G})$ coincides with the homology of the complex

$$
\mathcal{F}^{D} \otimes \mathcal{G} \to \mathcal{F}^{D} \otimes \mathcal{G} \otimes \mathcal{F}^{D} \otimes \mathcal{G} \otimes \cdots \to \mathcal{F}^{D} \otimes \mathcal{G}^{D},
$$

which is precisely $H$.

We shall now show how the Chern classes of certain line bundles vanish in $H^2(\mathcal{F}, \mathcal{G})$. In a subsequent section this leads to a generalised Bott vanishing theorem. We have to introduce some notation. We denote by $\mathcal{O}$ the sheaf over $\mathcal{X}$ of smooth complex valued function germs, and by $\mathcal{O}^{D}$ the subsheaf made up of those germs which are constant on leaves. Similarly $\mathcal{O}^{D}$ and $\mathcal{O}^{D}$ denote the (multiplicative) sheaves of non-zero complex smooth germs and those which are constant on leaves. We have now the exact sheaf sequences

$$
0 \to \mathcal{O} \to \mathcal{O}^{D} \to \mathcal{O}^{D} \to 0
$$

and

$$
0 \to \mathcal{O} \to \mathcal{O}^{D} \to \mathcal{O}^{D} \to 0
$$

(10)

where the first maps denote inclusions of the constant sheaf $\mathcal{O}$. Now also that only the sheaf $\mathcal{O}^{D}$ is fine in (10). Elements of $H^2(\mathcal{H})$ are called (equivalence classes of) smooth complex line bundles. See Hirsch [12] for the motivation for this terminology. By invariant line bundles we shall understand those lying in $H^2(\mathcal{H}, \mathcal{O}^{D})$ where the morphism is induced by the inclusion $\mathcal{O}^{D} \to \mathcal{O}^{D}$. The first Chern class $c_1(\mathcal{L})$ is defined for each line bundle $\mathcal{L} \in H^2(\mathcal{H}, \mathcal{O}^{D})$ as an element of $H^2(\mathcal{H})$ in the following way. The first of the sequences (10) gives us a long exact cohomology sequence, some of whose terms are

$$
H^2(\mathcal{H}, \mathcal{O}) \to H^2(\mathcal{H}, \mathcal{O}) \to H^2(\mathcal{H}, \mathcal{O}) \to H^2(\mathcal{H}, \mathcal{O})
$$

The two groups at the ends are zero, and the sheaf $\mathcal{O}$ is fine. The connecting isomorphism in the center is called $\alpha \cdot (-)$. If $\mathcal{L}$ is any sheaf containing $\mathcal{O}$, by the
first chern class in $\mathcal{A}$, we will understand the composition of the above morphism with the map $\Lambda^c(N_{\mathcal{A}}) \to \Lambda^c(N_{\mathcal{B}})$ induced by the inclusion $\mathcal{A} \subset \mathcal{B}$. For example to get real chern class one takes $\mathcal{B} = \mathcal{A}$.

**Proposition 3.** The first chern class of an invariant complex line bundle vanishes in $\mathcal{A}$.

**Proof.** We look at the following diagram:

\[
\begin{array}{ccc}
\Lambda^c(N_{\mathcal{A}}) & \to & \Lambda^c(N_{\mathcal{B}}) \\
\uparrow & & \downarrow \\
\Lambda^c(N_{\mathcal{B}}) & \to & \Lambda^c(N_{\mathcal{C}})
\end{array}
\]

(11)

where $\Lambda$ is the connecting homomorphism in the exact sequence induced by the second sequence in (10). The unnamed maps arise from inclusion. This diagram obviously commutes. The bottom row is zero due to exactness. So the proposition is proved if we can see that $\Lambda^c(N_{\mathcal{B}}) = \Lambda^c(N_{\mathcal{C}})$ is a monomorphism. Take a sufficiently small open cover $\mathcal{U}$ of $\mathcal{B}$. Suppose that we have a 1-cochain $\alpha \in \Omega^1_{\mathcal{A}}(\mathcal{U})$, on it whose co-boundary is in $\mathcal{B}$.\mbox{\ } \mathcal{U} = \mathcal{U}_j \cup \mathcal{U}_j$, where $\alpha_{12} \colon \mathcal{U}_{12} \to \Omega^0_{\mathcal{B}}(\mathcal{U})$ are smooth and constant on leaves. So we have

\[
\alpha_{12} = \alpha_{1} + \alpha_{2} + \alpha_{12}. 
\]

Since $\alpha_{12}$ is real we also have $\alpha_{12} = \alpha_{12} + \alpha_{12} + \alpha_{12}$, showing that the cochain $\alpha_{12} \in \Omega^1_{\mathcal{B}}(\mathcal{U})$ also has $\alpha_{12}$ as co-boundary.

We can complement the above proposition by including the converse statement, which follows immediately from the exactness of bottom row in (11).

**Proposition 4.** The first chern class of a smooth complex line bundle vanishes in $\mathcal{A}$ if and only if it is invariant.

In this form this proposition should be compared with a theorem of Inozemtsev, Kodera, Kodaira, Spencer and Delibeslitz-Thuven, 15.6.1. In irreducible which characterizes complex analytic line bundles over a complex manifold.

We have, as in (3), that $\Lambda^c(N_{\mathcal{B}}) = \Lambda^c(N_{\mathcal{C}})$

\[\Lambda^c(N_{\mathcal{B}}) = \Lambda^c(N_{\mathcal{C}}) \implies \Lambda^c(N_{\mathcal{A}}) \implies \Lambda^c(N_{\mathcal{B}}) \implies \Lambda^c(N_{\mathcal{C}}) \implies \Lambda^c(N_{\mathcal{D}}) \implies \Lambda^c(N_{\mathcal{E}}).
\]

In this decomposition, proposition 3 implies that the first real chern class of an invariant line bundle does not lie in $\Lambda^c(N_{\mathcal{B}})$. Equivalently if one looks at the diagram

\[
\begin{array}{ccc}
\Lambda^c(N_{\mathcal{B}}) & \to & \Lambda^c(N_{\mathcal{C}}) \\
\uparrow & & \downarrow \\
\Lambda^c(N_{\mathcal{C}}) & \to & \Lambda^c(N_{\mathcal{D}})
\end{array}
\]

(12)

where the inclusion $\Lambda^c(N_{\mathcal{B}}) \to \Lambda^c(N_{\mathcal{B}})$ results since $\Lambda^c(N_{\mathcal{B}}) = \Lambda^c(N_{\mathcal{B}})$ and $\Lambda^c(N_{\mathcal{B}})$ gotten from $\Lambda^c(N_{\mathcal{B}})$ by successively taking the kernel under the differentials $d_1, d_2, d_3, \ldots$ of section 5; then proposition 3 shows that the bottom row evaluates to
zero. The commutativity of the rectangle thus shows
that if $\mathfrak{g}$ is invariant $a_1(x)$ projects to zero under the
natural map $H^0_1(x) \rightarrow H^0_2(x)$. In other words $a_1(x) \in H^0_1(x)$
$H^0_2(x)$

We record this as a Corollary. If $\mathfrak{g}$ is invariant
$a_1(x) \in H^0_1(x)$. So it can be represented by a closed
form $\mathfrak{a}_1$. The converse statement is also true.

We remark that the rectangle in (12) commutes
for the following reason: The map $H^0(x) \rightarrow H^0_1(x)$
in this rectangle arises from the projection map
$(\mathcal{A}, \Delta) \rightarrow \left( \mathcal{A}_1, \Delta_1 \right)$. These $2$ complexes provide resolutions
of the two sheaves $\mathcal{A}$ and $\mathcal{A}$ which commute with the
inclusion $\mathcal{A} \rightarrow \mathcal{A}$.

$$
\begin{array}{c}
\mathcal{R} \\
\mathcal{B}
\end{array} \\
\downarrow \quad \downarrow
\begin{array}{c}
\mathcal{E} \\
\mathcal{D}
\end{array} \\
\begin{array}{c}
\mathcal{E} \\
\mathcal{D}
\end{array}

\xrightarrow{p} \hspace{0.5cm} \xrightarrow{q}
\begin{array}{c}
\mathcal{A} \\
\mathcal{A}
\end{array}
\xrightarrow{r}
\begin{array}{c}
\mathcal{A} \\
\mathcal{A}
\end{array}

\xrightarrow{s}
\begin{array}{c}
\mathcal{A} \\
\mathcal{A}
\end{array}
\xrightarrow{t}
\begin{array}{c}
\mathcal{A} \\
\mathcal{A}
\end{array}
\xrightarrow{u}
\begin{array}{c}
\mathcal{A} \\
\mathcal{A}
\end{array}
\xrightarrow{v}
\begin{array}{c}
\mathcal{A} \\
\mathcal{A}
\end{array}
\xrightarrow{w}
\begin{array}{c}
\mathcal{A} \\
\mathcal{A}
\end{array}
$$

2. In this section the question of homotopy invariance
of the spectral sequence shall be posed and solved.

Suppose that $\mathcal{N}_0 \rightarrow \mathcal{N}_0$ is a smooth map between two
foliated manifolds which leave leaves fixed, or,
to be precise is such that the induced map $\mathcal{N}_0 \rightarrow \mathcal{N}_0$
satisfies $\mathcal{f}(\mathcal{N}_0) = \mathcal{N}_0$. One now has another induced
map $\mathcal{N}_0 \rightarrow \mathcal{N}_0$ which is a vector space homomorphism
presenting the filtrations. Thus we have induced

homomorphism $\mathcal{N}_0 \rightarrow \mathcal{N}_0$, commuting with the dif-
ferentials $\mathcal{d}_0$. (see Cartan and Eilenberg). Suppose we
have two homomorphisms $\mathcal{f}(\mathcal{N}_0) \rightarrow \mathcal{N}_0$ and we can find a
chain homotopy between them, $\mathcal{d}_{\mathcal{a}} = \mathcal{d}_{\mathcal{b}}$ such that
$\mathcal{d}_{\mathcal{a}} = \mathcal{d}_{\mathcal{b}} = \mathcal{a} - \mathcal{b}$, then the two induced maps
$\mathcal{f}_0(\mathcal{N}_0) \rightarrow \mathcal{N}_0$ are identical. And the induced
maps $\mathcal{f}_0(\mathcal{N}_0) \rightarrow \mathcal{N}_0$ are the same if $\mathcal{f}$ is big enough
(section 3). Let us now put on the chain homotopy the
additional requirement that $\mathcal{f}(\mathcal{N}_0) = \mathcal{N}_0$ for all $\mathcal{f}$,
i.e., that the homotopy distorts the filtration by at
most $2$ units, then one can see from (3.2) p. 111, that
for $\mathcal{f} \neq 0$, the $2$ induced maps $\mathcal{f}_0(\mathcal{N}_0) \rightarrow \mathcal{N}_0$ are
identical.

Now the product $\mathcal{N}_0 \times \mathcal{I}$ can be foliated in two
natural ways. We'll say that it is $\mathcal{I}$-foliated if its
leaves are gotten by multiplying the leaves of $\mathcal{N}_0$ by $\mathcal{I}$.
And, we'll say that it is $\mathcal{I}$-foliated if its leaves are
just: (leaf of $\mathcal{N}_0$, $\mathcal{I}$). So in the first case the
cohomologies are unchanged, while in the second case
the dimension is $\mathcal{P}$-enlarged.

We say that $\mathcal{I}$ maps $\mathcal{I}(\mathcal{N}_0) \rightarrow \mathcal{N}_0$, which map leaves
into leaves are $\mathcal{P}$-enlarged ($k = 1, 2$). If we can find a
smooth map $\mathcal{N}_0 \times \mathcal{I} \rightarrow \mathcal{N}_0$ which takes the $k$-leaves into
leaves, with $\mathcal{N}_0 = \mathcal{I}$ and $\mathcal{N}_0 = \mathcal{E}_1$. 

Proposition 6. If $\xi_{k}, \eta_{k} \in H_{k}$ are $k$-homotopic (in the above sense) one can find a chain homotopy for which $\nu(N_{k}, \eta_{k}) = \nu(N_{k}, \eta_{k})$. (Such an $\nu$ is called a $k$-chain homotopy.)

As pointed out above this will have the following consequence.

Corollary 7. The spectral sequence is a $k$-homotopy invariant from the $H_{k}$ term onwards ($k = 1, 2$).

One recalls that the spectral sequence of a filtering is stable from the $H_{k}$ term onwards. The notion of fibre homotopy coincides with the notion of $k$-homotopy which has been introduced above. The next section deals with the proof of proposition 6. It involves a construction which will be helpful subsequently in building up a parametrix for $d$.

10. Proof of proposition 6. Given a vector field $X$ on the manifold, one has a skew-derivation $L_{X} : A \to A$ of degree $-1$ called the interior product with respect to $X$. We recall Kotschick's and Némusí [47], p. 33--

that if $\alpha$ is an $r$-form the definition of $L_{X}$ is

$$L_{X}(\alpha)(y_1, \ldots, y_r) = X\alpha(y_1, y_2, \ldots, y_r, y_r)$$

(13)

The property of the interior product needed to construct a chain homotopy is

$$L_{X} = d_{X} + i_{X}d$$

(14)

Now turning to the two given maps $\xi_{k}, \eta_{k} \in H_{k}$ and their homotopy $\lambda(N_{k}) \to H_{k}$ we have the induced morphisms $\xi_{k}(N_{k}) \to \lambda_{k}$ and $\eta_{k}(N_{k}) \to \lambda_{k}(N_{k})$ each preserving the filtration. Let us now take the vector field to be the lift of the standard vector field $\frac{\partial}{\partial t}$ on $T_{X}X$ is a vector field in $H_{k} \times X$. Let us define the morphism $\lambda(N_{k} \times X) \to \lambda(N_{k} \times X)$ by the formula (13). From this formula, and the fact that $\lambda_{k} \times X$ commutes with the $k$-foliation it follows that this map disturbs filtration by $k+1$ units only. Now we define one more homomorphism $\lambda(N_{k} \times X) \to \lambda(N_{k})$ in the following way. Let $\gamma : H_{k} \to H_{k} \times X$ by the map $x \to (X, x)$. Then $\int_{0}^{t} \gamma^{*} = \int_{0}^{t} (X, x)dt$.

Finally we define a chain homotopy $\lambda_{k} + \eta_{k}$ by the following composition

$$\lambda_{k} \to \lambda(N_{k} \times X) \to \lambda(N_{k})$$

(15)

It is clear that this map disturbs the filtration by $k+1$ units, and is of degree $-1$. The formula

$$d_{k} + \eta_{k} = \alpha \to \xi_{k}$$

follows by integrating (13):

$$d_{k} + \eta_{k} = \lambda_{k}^{*} = \int_{0}^{\xi_{k}} \gamma^{*} - \int_{0}^{\eta_{k}} \gamma^{*} - \int_{0}^{\eta_{k}} \gamma^{*}$$

as $d$ commutes with the induced map $\lambda^{*}$ and with $\int_{0}^{\xi_{k}}$

which was defined by the induced maps $\lambda_{k}^{*}$. So

$$\lambda_{k} \to \lambda(N_{k} \times X) \to \lambda(N_{k})$$

(15)
as $X$ was the lift of the vector $\frac{1}{2}x_1$ on $I$. 

11. We now study the behavior of the exterior product with respect to our filtration. Let us suppose that $u \in \Lambda^{p} I$ and $v \in \Lambda^{q} I$. Then the $p+q$ form $\omega \wedge \sigma$ is defined by

$$\omega \wedge \sigma(x_1, x_2, \ldots, x_{p+q})$$

where $\sigma$ is a permutation of the set $\{1, 2, \ldots, p+q\}$ and $\sigma(\sigma)\) denotes the parity of the same. Now suppose that $p+q-1-j+1$ of the vectors $x_1, x_2, \ldots, x_{p+q}$ lie in $D$. This implies that whenever $\leq p-1$ of the vectors $\lambda^{1}_{\lambda}, \ldots, \lambda^{p}_{\lambda}$ are in $D$, $\lambda^{q-j+1}_{\lambda}$ of the vectors $\lambda^{p+q}_{\lambda}$ are in $D$. Hence one of the two factors in each term of (16) is always zero. Thus $\omega \wedge \sigma \in \Lambda^{p+q}$. The product above defined is related to the exterior derivative in the following well-known way

$$d(\omega \wedge \sigma) = d\omega \wedge \sigma + (-1)^{p} \omega \wedge d\sigma$$

This fact is expressed by saying that $d$ is a skew derivation. From this fact it follows that a product.

the 'cup' product, is induced in $H(A)$; this product obeys the well known anticommutativity rule

$$ab = \{(-1)^{mp}a, b\}$$

If $a \in H^{p}(A)$, $b \in H^{q}(A)$, then $ab \in H^{p+q}(A)$ and $ab = \{(-1)^{mp}a, b\}$.

Proof: Immediately follows from definitions.

If $a \in H_{j}(A)$ and $b \in H_{j}(A)$, then $ab \in H_{p+q}(A)$. In particular, if $i+q = 0$, the codimension of the foliation, $ab = 0$.

This proposition puts strong conditions on the ring structure of $H(A)$: these will give "vanishing theorem" (see below). Note that we have thus a multiplication induced in the $R$ term of the spectral sequence. If $a \in R_{n} = H_{i}(A)$ has a representative

$$a \in H_{i}(A)$$

and similarly $b \in H_{j}(A)$ represents

$$b \in H_{j}(A)$$

then as $i \neq j$ is represented by

$$ab \in H_{i+j}(A)$$. It is clear that the choice of representatives is irrelevant. We shall denote this induced
product in \(E_n\) by \(a \wedge b\) also. Note that one may very well have non-zero \(e_i\) in \(H_i(a)\) and \(H_j(b)\) with \(ab \neq 0\) and still have \(a \wedge b = 0\); for, \(a \wedge b\) could lie in \(H_{i+j-1}^e(a)\) and we have \(H_{i+j-1}(b)\).

But if \(i + j \geq n\), one can conclude that \(ab = 0\) if \(a \wedge b = 0\) in \(E_n\). In an entirely analogous fashion the exterior product in \(E\) induces a product in \(E_0\) which pairs \(E_1^2 = E_2^1\) to \(E_0^1 = E_.\) The equation (7) leads to the equation \(d_h = d = 0\) for \(e \neq 0\) and \(e \neq 0\) when \(e \neq 0\) and \(e \neq 0\).

Hence this product in turn will induce a product in the homology of \(E_0\), via \(E\), and the differential \(d_h\) will be a skew derivation with respect to this induced product. The above remark holds at every stage; if \(i + j > n\) then vanishing in \(E_{i+j}\) implies vanishing in \(E_n\).

12. As a first example of exploiting the above proposition, we shall study the \(\text{signature}\) of a \(4k\)-dimensional oriented manifold \(M\). We recall the definition of the \(\text{signature}\). The cup product provides us with a bilinear symmetric \((2, 1)\) quadratic form on the vector space \(\text{II}^2(H, \mathbb{R})\) obtained by evaluating the product of any 2 classes in \(\text{II}^2(H, \mathbb{R})\) on the orienting \(4k\)-cycle of the manifold. As we denote this form by \(P, P(x, y) = x \cdot y\). The \(\text{signature of} P\) is called \(\text{sign} M\) and is a topological invariant.

**Proposition 5.** If \(n\) is odd, and \(\text{dim} \text{II}^{2k}\) \(\geq \text{dim} \text{II}^{2k}(1)\), then \(\text{sign} M = 0\).

**Proof.** Here \([\frac{n}{2}]\) denotes the first integer after \(\frac{n}{2}\).

We define the \(\text{cone} F\) of \(P\) as the subset of \(\text{II}^{2k}(H, \mathbb{R})\) given by the equation \(F(x, x) = 0\). It should be denoted by \(\gamma(P)\). Let us denote by \(p\) and \(q\) respectively the number of positive and negative values when \(F\) is reduced--by changing bases--to a diagonal form. It is well-known that that \(p\) and \(q\) are independent of the reduction process. And \(p - q = \text{sign} M\). By Poincaré duality we have \(p - q = \text{dim} \text{II}^{2k}(1)\), as \(1\) is a nonsingular quadratic form. Now we notice, by using Proposition 5, that \(\text{dim} \text{II}^{2k}(1) = \gamma(P)\). And therefore, since it is well known that either of the 2 nos. \(p, q\) is at least equal to the maximal dimension of subspace \(\gamma(P)\), we get the two inequalities \(p \geq \text{dim} \text{II}^{2k}\) and \(q \geq \text{dim} \text{II}^{2k}\). Using the given hypothesis it follows that \(p \geq \text{dim} \text{II}^{2k}\) and \(q \geq \text{dim} \text{II}^{2k}\). So it follows that \(p - q = \text{dim} \text{II}^{2k}\) and \(\text{sign} M = 0\). \(\Box\)
Some foliations obey a type of duality which resembles that observed by Serre [19] in complex manifolds. We'll call this Serre duality. It states that $L^k L^s = e^{-1} L^s$. Using this and (5) we see that the hypotheses of proposition 6 are satisfied.

Corollary 10: A 4k-dimensional oriented manifold admits a foliation with codimension k which satisfies Serre duality, only if the signature vanishes.

It is known that fibered manifolds in which the fundamental group of the base space acts trivially on the cohomology groups of the fiber obey Serre duality. (See [8].) Serre duality undoubtedly holds in other instances, but the work in this respect is as yet unfinished.

11. In this section we state and prove a generalized Bott vanishing theorem. We recall [13]—that $H^n(M; G)$ is defined, as a set with distinguished elements, even if $G$ is a sheaf of nonabelian groups. The important cases in this section are when $G$ is either the sheaf $G^a$ of smooth germs from $M$ into $G$, or else the sheaf $G^b$ of sections which are constant on leaves from $M$ into $G$. The elements of $H^n(M; G)$ are called smooth $G(n; G)$-bundles over $M$. But that given a space $Y$ and an action of $G(n; G)$ on $Y$, one can construct for each $f: H^n(M; G)$, a fibre bundle with group $G$ and fibre $Y$: simply take an open cover $Y$ in which a couple $g_i, i \in Y$ representing $G$ can be found and use the $g_i$ as coordinate transformations. Of course $G$ can be chosen as another Lie group, and the definitions hold. If $H = G$ then we have $H^b < G$ and $H^a = G$ and these sheaf inclusions induce morphisms $H^a(M, G) \to H^b(M, G)$ and $H^b(M, G) \to H^b(M, G)$ etc. We shall denote bundles lying in $H^a(M, G)$ by the name invariant G-bundles. And we shall say that the group of such a bundle can be invariantly reduced if the bundle lies in $H^a(M, G) \to H^b(M, G)$.

We note in passing that the group of a $G(n; G)$ bundle on $M$ can be always reduced to $U(n, G)$, but the group of an invariant $G(n, G)$ bundle on $M$ need not be invariantly reducible to $G(n, G)$. However for the bundle of transverse 1-forms, the group can be invariantly reduced from $G(n, G)$ to $G(n, G)$ if the manifold admits a $G(n, G)$-like metric. The concept of a $G(n, G)$-like metric for a foliation is due to Felder [22]. It is a Riemannian metric for which there exist coordinate systems such that $g_{ij}$, for $i, j > 1$, is constant on leaves.) We shall however not introduce any metrics; and the only
Invariant reduction which we encounter shall happen in a natural way soon. We remark again that the bundle of transverse 1-forms is invariant as we have local trivializations by invariant and transverse 1-forms, and the coordinate transformations of these are functions \[ U_j \times Y_j \rightarrow \text{Diff}(U_j) \] which are constant on leaves. (We recall Borel’s theorem of [12]—how characteristic classes can be defined using a theorem of Borel’s.) For a given \( \xi \in H^1(U, \mathbb{Z}) \) one constructs an associated principal bundle, i.e., the fiber as \( G \) and the action as left translation. We denote this bundle by \( P = \Pi(U) \). Hence we have an induced bundle \( \mathcal{P}^\xi \subset H^1(\mathbb{Z}/\mathbb{Z}) \). Here we denote by \( \Delta(n, 2) \), or just \( \Delta \), the subgroup of \( \text{GL}(n, 2) \) consisting of triangular matrices (i.e., elements below the diagonal vanish).

Proposition 1. The group of \( \mathcal{P}^\xi \) can be invariantly reduced to \( \Delta \).

Proof: If the word invariant is dropped this is a standard theorem from Steenrod [13]. The same proof works even now. We have a canonically given bundle with group and fiber \( G \); sitting over \( P/\Pi \) is \( \mathcal{P}^\xi \). Obviously since the coordinate transformations of \( P \) over \( K \) can be chosen constant on leaves, we can choose this bundle (note that \( P/\Pi \) is a compact smooth manifold. The fibration \( P/\Pi \rightarrow \mathbb{R} \) picks the foliation of \( \mathbb{R} \) up to a foliation of \( P/\Pi \). The fiber of this fibration \( P/\Pi \rightarrow \mathbb{R} \) is precisely \( \mathbb{R}/\Pi \), the manifold of leaves, each element of \( \mathbb{R}/\Pi \) is a sequence of subspaces \( 0 = \mathbb{R} \rightarrow \mathbb{R} \rightarrow \cdots \rightarrow \mathbb{R} \rightarrow \mathbb{R}^k \). Let \( \eta \in H^1(\mathbb{R}/\Pi, \mathbb{Z}) \) be associated to this bundle, then we assert that the induced map arising from \( \mathbb{R}/\Pi \rightarrow \mathbb{R} \) sends \( \eta \) to \( \mathcal{P}^\xi \). The proof of this fact can be found in Hirsch-Ne [15].

Now we resume our definition of characteristic classes. For each \( k = 1, 2, \ldots \), we have a map \( \Delta \rightarrow \mathbb{C}^k \) which picks out the \( k \)th diagonal element of the triangular matrix. It then induces a map \( H^k(U/\Pi, \mathbb{Z}) \rightarrow H^k(U, \mathbb{C}) \) and corresponding to \( \mathcal{P}^\xi \in H^k(U, \mathbb{C}) \) we get \( k \) complex line bundles \( \mathcal{P}^\xi \), all invariant. Thus one can check, as in [17], that \( \mathcal{P}^\xi \) in continuously isomorphic to the Whitney sum \( \mathcal{P}^\xi \oplus \cdots \oplus \mathcal{P}^\xi \). We remark that this is not an isomorphism as invariant bundles, however this is not needed. Now we define the Chern classes \( c(\mathcal{P}^\xi) \in H^{2k}(\mathbb{R}/\Pi, \mathbb{Z}) \) by the formula \( c(\mathcal{P}^\xi) = c(\mathcal{P}^\xi_1) \cup c(\mathcal{P}^\xi_2) \cup \cdots \cup c(\mathcal{P}^\xi_k) \) employing the cup product; the Chern class \( 1_{\mathcal{P}^\xi} \) of a line bundle having been defined already in section 8. Finally
we appeal to the theorem of Borel, which says that the projection $p_f^*|_X$ induces a monomorphism $\pi^*\mathcal{O}(\mathbb{A},\mathbb{Z}) \rightarrow \mathcal{O}(\mathbb{A},\mathbb{Z})$ to pull this class to $\mathbb{A}$. Thus $c(\mathbb{A}) = p^*c(\mathbb{A})$ -- the theorem of Borel also ensures that $c(\pi^*)$ lies in $\mathbb{Z}$.

**Proposition 13.** The real Chern ring of an invariant $\mathcal{O}(\mathbb{A},\mathbb{Z})$ bundle over $X$ vanishes in dimensions $> 2\pi$.

**Proof.** The fact that $\pi^*\mathcal{O}(\mathbb{A},\mathbb{Z}) \rightarrow \mathcal{O}(\mathbb{A},\mathbb{Z})$ is a monomorphism allows us to assume that the given invariant bundle $\pi^*\mathcal{O}(\mathbb{A},\mathbb{Z})$ can be invariantly reduced to $\mathbb{A}$. So it is the continuous Whitney sum of invariant line bundles $\xi = \xi_1 + \cdots + \xi_r$, $\xi_j \in \mathcal{O}(\mathbb{A},\mathbb{Z})$. Using corollary 5 of section 5, $\xi_j^2 = \chi(\mathbb{A})$. Then we use prop. 5 to conclude that an $r$-fold product of these classes will vanish if $r > 0$. This proves the above theorem.

We remark that unlike Prop. 4 we used the fact that $\mathbb{A}$ has a filtration of length $\pi$ in an essential way. If one filters (for a continuous foliation) the complex of singular cochains we are not sure that the spectral group $E^{r+1}_i$ vanish for $i > \pi$. So the above result is valid only for smooth foliation, whereas the vanishing theorem of prop. 4 is valid for topological

foliations.

Bott [4] proved this theorem, only for $\mathbb{A} = S^2$, and in a completely different way. This other method, which uses the Chern-Weil map, fits in naturally with our spectral sequence and shall be developed further in section 19.

Of course, as Bott and Husemoller [3] have pointed out, Theorem 9 breaks down for the integral Chern ring. The reason for the (real) Bott vanishing theorem can be traced back to the second of the exact sequences in (10), which gives us the exactness in the bottom row of (11).

We remark that one can build up a $K$-theory for invariant bundles; just as one has the $K$-ring of complex analytic bundles.

14. In this section, we suppose that we have chosen a complementary subbundle $N$, i.e., $D + N = C$, and $N$ projection maps $P_1, P_2, T$ with images $D$ and $N$ respectively. Then following a paper of Quenouille and Spencer [12] define for each pair $r, n$ such that $r + n = p$, a bundle map $P_{n+1}T \rightarrow P_nT$ as follows: if $U_1 \wedge U_2 \wedge \cdots \wedge U_p$ is a $p$-covector write it down as $(P_{n+1}U_1 \wedge P_nU_2 \wedge \cdots \wedge P_nU_p)$ and pick only
these terms in which $F_1$ occurs $r$ times and $F_2$ occurs $s$ times. We now think of $\Lambda^r$ as $(\Lambda^r)^*$ and so we have induced maps $F_{r+1}^* \Lambda^r = \Lambda^r$. The fixed points of this endomorphism form a subspace which we denote by $\Lambda_{r,s}^r$. From the definition of $R$, via $\Lambda^r_{r,s}$ it follows that we have an isomorphism $\Lambda^r_{r,s} \cong \Lambda^r_{r,s}$.

$\Lambda^r_{r,s} \cong \Lambda^r_{r,s}$. We have some more simple relations: $\Lambda^r = \bigotimes \Lambda^r_{r,s}, \Lambda^s = \bigotimes \Lambda^r_{r,s}, \Lambda_1 = \bigotimes \Lambda^r_{r,s}$ etc.

Now our filtration is preserved by the exterior-derivative, $d(\Lambda) = \bigotimes \Lambda^r_{r,s}$. It follows that $d(\Lambda^r_{r,s}) = \Lambda^r_{r,s}$ and $d(\Lambda^s_{r,s}) = \Lambda^s_{r,s}$ with $\Lambda^r_{r,s}$ and $\Lambda^s_{r,s}$ with $\Lambda^r_{r,s}$ respectively. Furthermore, the endomorphism $d$ is a skew derivation with respect to the exterior product. It follows from these remarks that in the bigraded module $\Lambda = \bigotimes \Lambda^r_{r,s}$, $d$ can be thought of as the sum of three endomorphisms $d_1, d_2$, and $d_3$ of degrees $(0,1)$, $(1,0)$, and $(2,0)$ respectively. Now the equation $d^2 = 0$ can be written.

In other words $d$ becomes the sum of three endomorphisms, each of order 2, and any two of these commute up to sign.

**Proposition 13.** $d_{r,s} = 0$ if and only if $N$ is involutive.

**Proof.** Follows immediately from Proposition 1 applied to the filtration gotten from $N$ in place of $O$. QED

Using this bigrading we can algebraically characterize $E_1$ and $E_3$. The endomorphism $d_{1,1}$ of order 2 acting on the above bigraded space $\bigotimes \Lambda^r_{r,s}$ gives us its homology which we denote by $H_{r,s}(A)$. Again since $d_{1,1}$ anti-commutes with $d_{0,1}$ it induces a differential on this new bigraded space $\bigotimes H_{r,s}(A)$ of degree $(1,0)$.

If we take homology with respect to this differential we get a new bigraded space $H_{1,0}H_{r,s}(A)$. Similarly we have yet another bigraded space $H_{1,0}H_{r,s}(A)$.

**Proposition 14.** We have $E_1 = H_{r,s}(A)$ and $E_3 = H_{1,0}H_{r,s}(A)$.

**Proof.** The proof follows that in Cartan and Eilenberg (p. 330). The seventh differential in our spectral sequence, $d_{7,1}$, is given by $d_{7,1}^* = \frac{1}{2} d_{3,1}^* d_{4,1}^* + \frac{1}{2} d_{4,1}^* d_{3,1}^* - \frac{1}{2} d_{5,1}^* d_{2,1}^* - \frac{1}{2} d_{2,1}^* d_{5,1}^* - \frac{1}{2} d_{1,1}^* d_{4,1}^* - \frac{1}{2} d_{4,1}^* d_{1,1}^*$. The sum of these two is zero. It should notice this is the $d_{0,1}$ above. So they coincide and we get $E_1 = H_{r,s}(A)$. Now see Ol p. 119—the first differential in our spectral sequence.
sequence $A_1^2$ to $A_2^1$ can be seen to be the same as the connecting homomorphism induced by the following exact sequence: 
$0 \to A_{g+1} \overset{A_{g+1}}{\to} A_{g+2} \to A_{g+1}^{g+1} \to 0.$
\[ \text{Note, from (4'), that } E_1 \to E_2 \text{ is same as } \partial \left( \frac{A_1^{g+1}}{A_2^{g+1}} \right). \]
Now the connecting homomorphism $\partial \left( \frac{A_1^{g+1}}{A_2^{g+1}} \right)$, given as we pass to homology under $d$, is obviously the same as the map induced by $d_{g+1}$ which is the part of $d$ having degree $1$, (as $d_{g+1}$ would take us to $A_{g+2}$ and thus play no role in above connecting morphism). So it enables us to identify $d_1$ and $d_{g+1}$ and see that $d_2$ is same as $h_2$ (as $\mathcal{H}_1$).

It is however not true that $E_2 = \mathcal{H}_1 \otimes \mathcal{H}_2$ in the spectral sequence as the differential $d_3$ of the spectral sequence is quite different from the above morphism.

13. The rather algebraic interpretation of the $E_2$ term, given by Prop. 14 is not altogether satisfactory.

A more geometric result is the following.

**Proposition 15.** The sheaf sequence
\[ 0 = L^q \to \mathcal{H}^q_{g+2} \to \mathcal{H}^q_{g+1} \to 0 \quad (19) \]
is exact. For each $q \geq 0$, we have the induced sheaf complex
\[ 0 = \mathcal{H}^q_{g+2} \to \mathcal{H}^q_{g+1} \to \mathcal{H}^q_{g} \to \mathcal{H}^{q+1}_{g+1} \to \mathcal{H}^q_{g+2} \].

Under the isomorphism of prop. 3 this complex is same as
\[ 0 \to \mathcal{E}^q_{g+2} \to \mathcal{E}^q_{g+1} \to \cdots \to \mathcal{E}^q_2 \to \mathcal{E}^q_1 \to 0. \]
Thus the $E_2$ term can be thought of as the homology of the seqn. (20).

Proof: First we demonstrate the exactness. To do this we employ the classical Poincaré's lemma in the following way: a local section of $\mathcal{E}^q$ is a local invariant transverse $p$-form $\omega$ looks like
\[ \Sigma \omega_1 \partial x_1 \wedge \cdots \wedge \partial x_p \text{ in coordinates } x_1, x_2, \ldots, x_p \]
in $\mathcal{E}_1 \cdots E_p$ compatible with the foliation, and $\mathcal{E}^q(\omega)$ is just
\[ \Sigma \partial \omega_1 \wedge \partial y^1 \wedge \cdots \wedge \partial y^p. \]
Using the fact that in $\mathcal{E}^0$ any closed form is exact, we are through.

For the second part we recall that the isomorphism of prop. 3 resulted from the finite sheaf resolutions occurring in the rows of the following long-exact diagram of sheaves:
\[ 0 \to \mathcal{E}^1_{g+2} \to \mathcal{E}^1_{g+1} \to \cdots \to \mathcal{E}^1_2 \to 0 \]
\[ \mathcal{E}_{g+1} \to \mathcal{E}_{g} \to \cdots \to \mathcal{E}_{1} \to 0 \]
\[ 0 \to \mathcal{E}^1_{g+2} \to \mathcal{E}^1_{g+1} \to \cdots \to \mathcal{E}^1_2 \to 0. \]
Here I think of the sheaf $\mathcal{O}^{\mathbb{R}}_\mathbb{R}$ as the sheaf of germs of forms in $\mathbb{R}^{\mathbb{R}}$ (see Section 14 above). Only the commutativity of the first square could be non-obvious. It follows by noting that an invariant forms, $\omega_{\mathbb{R}} = 0$. And since $\mathcal{K}^{\mathbb{R}}_\mathbb{R} = \mathcal{K}^{\mathbb{R}}$, it is obviously zero for $q = 0$. We see that $d\omega^{\mathbb{R}} + \omega^{\mathbb{R}}$ is same as $\omega_{\mathbb{R}}^{\mathbb{R}} + \omega^{\mathbb{R}}$, and so the first square commutes.

Due to naturality, the second assertion follows from this commutative diagram.

In a well-known special case, here [31] was able to give a better description of the spaces $\mathbb{R}_\mathbb{R}^\mathbb{R}$. We shall now obtain his results. So we suppose that our foliation arises from a smooth fibration $F \rightarrow B$ with fiber $F$. Here $F$ and $B$ are smooth manifolds, etc. Before taking up the general case we note that the case $q = 0$ is easy.

**Corollary 15.** In this foliation case $\mathbb{R}_\mathbb{R}^\mathbb{R} = \mathbb{R}_\mathbb{R}^\mathbb{R}(F,B)$. Proof: For $q = 0$, (20) is just the chain complex of sections arising from the differential form $\omega^{\mathbb{R}}$. Now the sheaf $\mathcal{O} = \omega^{\mathbb{R}}$ is simply the pull-back of the sheaf $\mathcal{O}(\mathbb{R})$ of 1-forms on the base space. Hence (20) coincides with the salient complex on $B$ and the result follows.

For the general case we define a sheaf $\mathcal{H}(F)$ on $B$ from the following pre-sheaf $\mathcal{H}(F)$ on each open set $U$ of $B$ we associate the vector space $\mathbb{R}(\mathbb{R}^\mathbb{R}(F,U))$, and to each inclusion map $N \subseteq U$ the induced homomorphism $\mathcal{H}(\mathbb{R}^\mathbb{R}(F,N)) \rightarrow \mathcal{H}(\mathbb{R}^\mathbb{R}(F,U))$. Now we have the homology of $B$ with coefficients in this sheaf, viz., $\mathcal{H}(\mathbb{R}^\mathbb{R}(F,B))$. It is usual to call this as the homology of $B$ with local coefficients in $\mathcal{H}(F)$. We then have the following:

**Proposition 17.** In the foliation case $\mathcal{H}(F,B) \cong \mathcal{H}(\mathbb{R}^\mathbb{R}(F,B))$.

Proof: We extend the construction above defined to the entire sequence (19). i.e., we construct sheaves $\mathcal{H}(\mathbb{R}^\mathbb{R}(F,H))$ on $B$ from the pre-sheaves which attach to each open set $U$ of $B$ the space $\mathbb{R}(\mathbb{R}^\mathbb{R}(F,H))$, and to each inclusion $N \subseteq U$ the induced map $\mathcal{H}(\mathbb{R}^\mathbb{R}(F,N)) \rightarrow \mathcal{H}(\mathbb{R}^\mathbb{R}(F,U))$. Now the morphism in the sequence (19) induce sheaf homomorphisms $\mathcal{H}(\mathbb{R}^\mathbb{R}(F,H)) \rightarrow \mathcal{H}(\mathbb{R}^\mathbb{R}(F,H))$. The resulting sheaf sequence

$$0 \rightarrow \mathcal{H}(\mathbb{R}^\mathbb{R}(F,H)) \rightarrow \mathcal{H}(\mathbb{R}^\mathbb{R}(F,H)) \rightarrow \cdots \rightarrow \mathcal{H}(\mathbb{R}^\mathbb{R}(F,H)) = 0$$

is exact. Moreover each of the sheaves $\mathcal{H}(\mathbb{R}^\mathbb{R}(F,H))$ is fine as follows from noting that $\mathcal{H}(\mathbb{R}^\mathbb{R}(F,H))$ is a module over $\mathcal{O}(\mathbb{R})$, the ring of functions constant on leaves. Finally the chain complex of sections arising from (21) coincides with (20). This proves the assertion.
We shall not pursue this special case further as it is well known and understood.

It should be pointed out however that the exact short sequence (19) is the basic reason why one uses spectral sequences to study foliated manifolds. In the terminology of Swan [35] the "second" spectral sequence of (19)—with resolution by forms—is precisely the spectral sequence being studied. One can generalize this procedure to studying any structure on \( \mathcal{N} \), which gives us from the exact sequence

\[
0 \to \mathcal{F} \to \mathcal{L}^k \to \mathcal{L}^{k-1} \to \cdots \to \mathcal{L}^0 \to 0
\]

of sheaves another (semi-exact sequence) of subsheaves.

\[
0 \to \mathcal{G} \to \mathcal{L}^k \to \mathcal{L}^{k-1} \to \cdots \to \mathcal{L}^0 \to 0
\]

In our case \( \mathcal{G} \) is the foliate structure and the subsheaf \( \mathcal{L}^k \) is \( \mathcal{L}^k \), the sheaf of transverse and invariant \( k \)-forms, and this sequence is the exact sequence (19).

(This general point of view can be seen in Spencer's work—see, e.g., [33].)

Let \( \mathcal{N} \) be any smooth manifold, not necessarily compact, and suppose that we have a smooth vector bundle \( \mathcal{E} \) over \( \mathcal{N} \); the space of smooth sections is denoted by \( \mathcal{C}^\infty(\mathcal{E}) \). By a local trivialization of this bundle we mean a diffeomorphism \( \phi \) of \( \mathcal{E} \) with the trivial bundle \( \mathcal{N} \times \mathbb{R}^k \). (Here \( \mathbb{R}^k \) is an open set in \( \mathbb{R}^k \) and \( k \) is the fiber dimension.) If we look at a section \( g \in \mathcal{C}^\infty(\mathcal{E}) \) under this trivialization we get a map \( \mathcal{E} \to \mathbb{R}^k \), whose kth coordinate shall be denoted by \( x_i^g , \quad 1 \leq i \leq k \). If \( m = (a_1, a_2, \ldots, a_n) \) is a multi-index we shall denote by \( \partial^{(m)} \) the function \( \mathcal{N} \to \mathbb{R}^k \) given by

\[
\partial^{(m)} = \frac{\partial^{a_1}}{a_1!} \frac{\partial^{a_2}}{a_2!} \cdots \frac{\partial^{a_n}}{a_n!}
\]

(Here \( \partial^{x_i} = \frac{\partial}{\partial x_i} = a_1 + a_2 + \cdots + a_n \).

Now choose a compact set \( \mathcal{L} \) over \( \mathcal{N} \) and set \( \| g \|_{\mathcal{L}} = \sup \langle \partial^{(m)}(x), g(x) \rangle \). Then this is a semi-norm on the vector space \( \mathcal{C}^\infty(\mathcal{E}) \). (A function \( p \in \mathcal{C}^\infty(\mathcal{E}) \) such that \( p(x + y) \leq p(x) + p(y) \), \( p(\lambda x) = \lambda \cdot p(x) \) is called a semi-norm on the vector space \( \mathcal{E} \).) We put on \( \mathcal{C}^\infty(\mathcal{E}) \) the weakest topology which makes all the semi-norms \( \| \cdot \|_{\mathcal{L}} \) continuous. One can in fact get this same topology by using only a countable number of these semi-norms. (Take a countable number of compact sets covering \( \mathcal{N} \) with their interiors, each compact set lying in a \( \mathcal{L} \) over which a trivialization is given.)

Also, all the semi-norms vanish only on the zero section. Hence this vector space is non-void. [See p. 26 of L. Schwartz [36]. We'll refer to this book for all results on functional analysis.]
preceeding sections we have come across a number of vector spaces of this type. For example, $L^2$, $L^p$, $\mathbb{R}^n$ are all spaces of sections of suitable vector bundles on $N$. These vector spaces, or for that matter any other such space of sections, shall be assumed to be topologized in the above manner.

**Proposition 15.** $C^0(V)$ is a Fréchet space.

(A Fréchet space is a complete metrizable space. By complete we mean that every Cauchy sequence converges.

A sequence $v_j$ is called Cauchy if $p(v_j - v_k) \to 0$ as $i,j \to \infty$ for each semi-norm $p$.)

Proof. We have already seen that its topology can be defined by a countable number of semi-norms. The completeness follows by noting that uniform limit of continuous functions is continuous.

**Corollary.** $L^p$, $L^p(E)$, $C^0(V)$ are Fréchet spaces.

We remark that a is a Hausdorff locally convex topological vector space--briefly, an LCV space--and, as such the theory of compact operators applies to it.

We recall that if $E$ and $F$ are $k$-algebras then a linear map $E \to F$ is called compact if it is continuous and maps some neighborhood of zero onto a relatively compact set (i.e., a set with compact closure). The following basic theorems in the calculus of the effects of Fredholm, Hilbert, Riesz and Schwartz amongst others.

**Proposition 16.** If $E$ is a Hausdorff locally convex topological vector space and $S - A$ is compact then the map $1 - A^2B - S$ has a finite dimensional kernel, a closed image, and a finite dimensional cokernel.

We refer the reader to Seno--[40], for general results. We shall need only the finiteness of codimension.

The volume bundle $\pi : N \to \mathbb{R}^n$ is the line bundle associated to the tangent bundle by the representation $GL(n) \to \mathbb{R}^n$ given by $A \mapsto \det A$. It is clear that sections of $\pi$ are smooth sections on $N$. A smooth kernel $K$--see Atiyah and Bott, Liebmerial Plancherel, formula [42]--assigns for each point $(x,y)$ of $N \times N$ a linear transformation $K_{xy} : L^p(\Omega) \to L^p(\Omega)$.

Thus for a given $x$, $K(x,y)(x)\in L^p(\Omega)$ is a form at $x$ times a measure. Integrating over $y$ we shall get a form which we shall denote by $K_2$. This integration is possible if $x$ is a compact support in $N \times N$; or, even if, for $x$ fixed, the set $\{y(x,y) \neq 0\} \subset N$ has a compact closure. A linear map $\mathbb{R}^k \to \mathbb{R}$ which arises from a smooth kernel $E$ in the fashion described is
called a smoothing map. We will write
\[ (\phi_w, y) = \frac{1}{\|y\|} \int_k \|y\| (\phi_w(y)) \]

We note that the definition of smoothing map holds even if we are working with an arbitrary vector bundle \( V \) in place of \( H^n \). Thus one can talk of smoothing maps \( \mathcal{C}^n(V) \to \mathcal{C}^n(V) \). The right side of (22)

continues to make sense even if \( y \) is only continuous in \( y \). Differentiation under the integral sign shows that

If \( K \) is still a smooth kernel, \( \Sigma_y \) shall be smooth in \( x \). In other words we have a natural extension

\[ \mathcal{C}^n_y(V) \to \mathcal{C}^n_y(V) \]

of \( x \)-times continuously differentiable sections of a vector bundle \( V \to K \) by the semi-norms \( \| \phi \|_n \), with

\[ \| \phi \|_n = \sup_{K} \| \phi \|_n \]  

It is apparent that if \( K \) is compact then the topology is given by a finite number of such semi-norms; we take a finite number of compact sets \( K \), whose interiors cover \( K \) and a finite number of trivializations \( \chi \) on neighborhoods of \( K \).

Thus the topology is also given by a norm

\[ \| \cdot \|_1 = \sup_{L, L} \chi, L, L \| \cdot \|_1 \]

The space is clearly complete. Hence \( \mathcal{C}^n_y(V) \) has been topologized, for \( K \)

compact, as a Banach space.

Proposition 22. If \( K \) is compact \( \mathcal{C}^1(V) \to \mathcal{C}^n(V) \) is compact.

Proof. Take any bounded neighborhood of zero in \( \mathcal{C}^n(V) \).

As a subset of \( \mathcal{C}^n(V) \) \( K \) is equicontinuous and fiber-wise bounded. [By Schwartz’s theorem, see, e.g.,

Schwartz, Th. 4-6, it has compact closure in \( \mathcal{C}^n(V) \).]

This result shows that the inclusion \( \mathcal{C}^n_y(V) \to \mathcal{C}^n_y(V) \) is also compact, as it factors through the above map. Hence we see that for \( K \) compact, a smoothing map \( \Sigma_y \to K \) is always compact.

By a parametric form—or simply, a parametric—

we shall understand, as in [2], a linear map \( \Lambda, L \) such that \( dp \times dp - 1 \) where \( \Lambda \Lambda L \) is a smoothing map.

When \( K \) carries a foliation we will also introduce a

further refinement: a parametric will be called a

\[ \text{2-parametric} \]

if it commutes with the filtration and

\[ \| \phi \|_1 \| \phi \|_1 \]

[One can always construct a parametric for \( d(\phi \cdot \pi) \), sections 17 and 18 below] but it is

not known whether a \( 2 \)-parametric is possible. An

example due to Schwartz [25] implies that one need

not have a 2-parametric.]

The important equation

\[ \lambda = a + dp + dp \]

shows that 1 = a maps \( \Sigma (\phi = a \wedge \Lambda, dp = 0) \) into \( \mathcal{C}^n \).
The author feels that the hypotheses in (a) are not required for this finiteness result. However, one would have to use more refined functional analysis to settle this point.

This proposition tells us that it is a good idea to construct $k$-parametrices for $d$. (This will be done in subsequent sections.) For example, a 2-parametrix would make $E_k$ finite dimensional under the above hypotheses. Note however that $E_k^{(j)}$ are always Hausdorff and the existence of a 2-parametrix would at once imply that $E_k^{(j)}$ are finite dimensional. Another simple observation is that $E_k$ is finite dimensional for $k$ compact—-for $k > 1$. This follows from the fact that $E_k$ is finite dimensional and that $E_k^{(j)} = 0$ for $k > 1$, as the filtration is of length $a$. We can ask for whether there exist $a_i$, independent of $a$, for which $E_k(a_i)$ is finite dimensional; $H$ being any compact foliated manifold. The recent counterexample of J. Schwartz [28] shows that if such an $a_i$ exists, it must be $a_i = 0$. This follows from the groups in (28) are precisely $E_k^{1/2}(H)$ and Schwartz gives foliations for which they fail to be finite dimensional. Note that [28] thus implies that one may not be able to construct a 2-parametrix.
we shall put
\[ e_{p,q} = \dim H_{p,q} \] (24)
if the right side is a finite number.

**Proposition 21.** If \( f \) is finite dimensional,
\[ x(k) = \sum (-1)^{mp} b_{m,p} \] (25)
where \( x(k) \) is the Euler-Poincare characteristic of \( K \).

**Proof.** Since the \( i \) th Betti number of \( K \) is
\[ b_i = \sum_{p,q} e_{p,q} \] for \( i = 1 \), it follows that
\[ x = \sum (-1)^{mp} b_{m,p} \] for \( i = 1 \),
and so
\[ x = \sum (-1)^{mp} b_{m,p} \] for \( i = 1 \),
for a large enough.

But we know that the Euler characteristic of a finite complex is same as that of its graded homology. Hence the last expression equals
\[ x = \sum (-1)^{mp} b_{m,p} \] QED

One can pose some general index problem: *e.g.* calculate \[ \sum (-1)^{mp} b_{m,p} \] (if \( f \) is finite dimensional) in terms of characteristic classes. It is possible in some cases to guess at the probable expressions. But the general development in this direction is at the moment held up due to analytical difficulties: construction of parametrizing etc., which we will encounter in the following sections.

If we have a Hausdorff space \( V \), equipped with a continuous differential \( \delta V = \delta \) then it is clear that \( H(V) \) is finite dimensional. Hence the continuous homology of \((V, \delta)\) is finite dimensional. We denote it by \( H(V) \).

Note that \( H(V) \) will be Hausdorff since. On the other hand the homology group \( H(V) \) need not be Hausdorff.

**Proposition 22.** The continuous homology of the chain complex is same as the dual complex homology. *Proof.* In fact \( B = \delta (x) \) can be characterized by duality theorem--as those forms which have zero value on all cycles. Thus it will follow that \( B = E \) and the result is clear. QED

Note that if \( K \) is compact, prop. 23 is trivial: \( B \) is of finite codimension in the Preset space \( K \), so it may be closed.

This proposition and prop. 21 show that it might be desirable to replace the spectral sequence \( E_p \) by a continuous spectral sequence (in which each step we take the continuous homology with respect to a continuous differential). The advantage of such a change would be that the Hausdorffness requirement can be dropped from prop. 21. If there is a hypersurface, \( E_p \) is finite dimensional by prop. 23 such a spectral
sequence would also converge to $\mathcal{H}(\mathbb{R}^n)$. However we shall not go into this here because (1) a lot of preliminary continuous homological algebra is required; and (2) the induced algebras condition ought to be studied, since it has connections with some duality, and not avoided. (See prop. 27 below.)

12. Let us denote by $\mathcal{C}^m(\mathbb{R},k)$ the set of all smooth maps $\mathbb{R} \to \mathbb{R}^m$, and by $\mathcal{C}^m(\mathbb{R},k)$ the set of all smooth maps $\mathbb{R} \to \mathbb{R}^m$ which map any leaf into another. Consider a function

$$F(x) = \mathcal{C}^m(\mathbb{R},k), \text{ with } F(0) = \text{id}.$$  \hspace{1cm} (25)

Let $f(t)$ be any function on $\mathbb{R}$ which is smooth, has compact support, and for which

$$\int_{\mathbb{R}} f(t) \, dt = 1.$$  \hspace{1cm} (27)

Let us now write the formal expressions, with $u \in \mathcal{C}^m(\mathbb{R},k)$,

$$(u)(x) = \int_{\mathbb{R}} \left( f(x, t) u(t) \right)(x') \, dt$$  \hspace{1cm} (28)

and

$$(u)(x) = \int_{\mathbb{R}} \left( f(x, t) u(t) \right)(x') \, dt.$$  \hspace{1cm} (29)

Here the map $F(\mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ is defined by

$(x, t) \mapsto \phi(x, t)$, and for each $t \in [0,1]$ we define the

Injection $I_4 : \mathbb{R}^4 \to \mathbb{R}^4$ by $x \mapsto (x, t)$. $\frac{\partial}{\partial x}$ is the standard vector on $\mathbb{R}^m$ along the $t$-direction. One should compare these formulæ to those in Section 10.

We can topologize the set of all maps, $\mathcal{C}^m(\mathbb{R},k)$, in a natural way with the $C^m$ topology, as such maps are 'near' each other if (in local coordinates) all their derivatives are 'near' each other. The subset $\mathcal{C}^m(\mathbb{R},k)$ shall be given the subspace topology.

Now if we require that the function $F$ be continuous in (26) it follows immediately that the two integrands in (28) and (29) are continuous. Since they have compact support also both these expressions make absolutely good sense.

**Proposition 26.** $p$ is a chain homotopy between $i$ and $u$, i.e., $1 - u = dp + dp$. If the image of (26) lies in $\mathcal{C}^m(\mathbb{R},k)$ then $a$ preserves filtration and $p$ is a $2$-chain homotopy.

**Proof.** Since $d$ commutes with induced maps we see that

$$d(a) = \int_{\mathbb{R}} \left( i + f(t) u \right)^{-1} \frac{\partial}{\partial x} (i(t) u) \, dt$$

equals

$$\int_{\mathbb{R}} \left( \phi(t, 0) + f(t) \phi(t, 1) \right) \frac{\partial}{\partial x} (i(t) u) \, dt.$$
which, by (14), is the same as
\[ \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \left( \frac{\partial \gamma}{\partial t} \right)^2 = -1 \, \text{d}t \] Integrating with respect to $t$ we get, as $P(0) = 1$
\[ \int_{\mathbb{S}^1} \, \text{d}t = 1 \] and so, by (27) and (28) this is the same as
\[ \omega = -\omega \]

Now, if $P(n) : M \to M$ maps leaves into leaves then $P(n) : E$ preserves the filtration. So, by (28) $\omega_{\mathbb{S}^1} = \omega$ preserves the filtration. Considering $M \times \mathbb{R}$ to be carrying the $\mathbb{S}^1$-foliation (section 9) we see that $P(n)^* \omega_{\mathbb{S}^1} = \omega_{(\mathbb{S}^1 \times \mathbb{R})}$ also preserves the filtration. Since $\Lambda_{\mathbb{S}^1}$ is transverse to the $\mathbb{S}^1$-foliation of $M \times \mathbb{R}$ it follows that the filtration of $\Lambda_{\mathbb{S}^1} \cdot P(n)^\ast \omega$ is one of $\mathbb{S}^1 \times \mathbb{R}$ unit less at most. Since $\omega \in \Lambda^2(M \times \mathbb{R}) \in \Lambda^2(M)$ preserves the filtration as does one from (29) that $\omega \in \Lambda^2$ obeys the condition $P(n)^* \omega = \Lambda_{\mathbb{S}^1} \cdot P(n)^\ast \omega$. 

Consider the following situation: $N$ is parallelizable, and admits a global parallelization by $n$ complete vector fields. This means that we have a globally defined tangent vector field on $N$, which is linearly independent at each point, and which define $1$-parameter groups of diffeomorphisms of $N$. Thus we have a continuous function $F: \mathbb{S}^1 \to \text{Diff}(N)$, $F(0) = 1$.

Now choose an $x \in M$ and consider the map $P_x$ of $\mathbb{R}^n$ into $M$ given by $(t \cdot x)$, where $t \in \mathbb{R}$. This map takes $0$ to $x$.

Also it maps $\mathbb{R}^n$, the tangent space to $\mathbb{R}^n$ at $0$ isomorphically onto $\mathbb{R}^n$. In fact, using the canonical identification of $T_x \mathbb{R}^n$ with $\mathbb{R}^n$, this map is actually the map $P_x : T_x \to T_x$ given by $t \cdot \gamma(t)$, and, by hypothesis,
It is an isomorphism. It follows therefore that \( F_2(x,x') = r \) maps some neighborhood of \( 0 \in \mathbb{R}^n \) diffeomorphically onto a neighborhood of \( x \in \mathbb{R}^n \). (In general we will say that the continuous map of (26) is \textit{locally} transitive at \( x \in \mathbb{R}^n \) if \( P_F(x,x') = r \) has the above property.)

Let us consider also the product space \( x \times N \) and let \( a \) denote the diagonal in this space. The map \( V \) leads to a natural map \( s = \eta^{-1}F_2^{-1}r^{-1}F_2r^{-1}F_2^{-1}r^{-1}F_2r^{-1} + (x,0) \) from \( (a,x') \) to \( (x,x') \). Again it follows from the given hypothesis that the tangent space at the first point \( (a,x') \) is mapped diffeomorphically to that at \( (x,x') \). Hence some neighborhood of the tangent space at the first point \( (a,x') \) is mapped diffeomorphically onto a neighborhood of the second. From this we conclude that given \( x \in x \times N \) one can find a neighborhood \( V \) of \( x \) such that a neighborhood of \( x \) in \( \mathbb{R}^n \) is mapped diffeomorphically onto some neighborhood of \( x \) by \( F_2 \) for each \( y \in V \). (This statement makes sense for any continuous map \( y \rightarrow \mathbb{R}^n \) and \( x \rightarrow \mathbb{R}^n \).) We will say then that \( F \) is \textit{locally} transitive at \( x \in N \).

Now if \( N \) is compact then the last sentence can be strengthened to read: There exists a neighborhood \( V \) of \( 0 \in \mathbb{R}^n \) which is mapped diffeomorphically to some neighborhood of \( y \in N \) by any \( F_2 \). \( F \in \mathbb{R}^n \). (We will say that (26) is \textit{uniformly} transitive if it possesses the property expressed by this sentence.)

We will assume now that the function \( f(x) \) used above in constructing \( s \) and \( p \) has its support inside the aforementioned neighborhood \( V \) of \( 0 \in \mathbb{R}^n \).

**Proposition 21.** If the continuous map \( F(x) = C^\infty(\mathbb{R},\mathbb{R}) \) is uniformly transitive and if \( f(x) \) has sufficiently small support, then \( s \) is a smoothing map, no \( p \) is a parametric in Prop. 20. If further the image of \( F \) lies in \( C^\infty(\mathbb{R},\mathbb{R}) \) then it is a 2-parametric.

**Proof.** By sufficiently small support we simply mean that the assumption just made above holds. Now \( s(1,\overline{s},0)(x) = x \) gives us a neighborhood of the diagonal \( x \in N \times \mathbb{R}^n \) such that for each \( (x,y) \) \( x \) we have a smooth linear transformation \( \mathbb{R}^n \rightarrow \mathbb{R}^n \) given by \( x \rightarrow \overline{s}(x) \). Let \( \mathbb{R}^n \) be the measure in \( \mathbb{R}^n \) corresponding to the measure \( \mathbb{R}^n \) on \( \mathbb{R}^n \) under the diffeomorphism \( \mathbb{R}^n \). Defining \( \mathbb{R}^n \) \( x \rightarrow \overline{s} \) we thus set a smooth kernel \( x \) on \( \mathbb{R}^n \). It is clear that the \( \mathbb{R}^n \) given by eqn. (25) is the same as \( \mathbb{R}^n \), a smoothing map, given by (26).
vector fields of the foliation, then we will get a $\mathbb{R}$-parameter. If $M$ is not compact, but still has a parallelism by complete vector fields (39), then $\pi$ is only locally transitive near each $x$. This situation will arise later on when we take $M$ to be a principal tangent bundle. However, in that case $F$ is equivalent with respect to group action on the fibers. Due to this $F$ will be uniformly transitive if the base space is compact. (This construction will be postponed to see if since it uses connection theory.)

10. In this section we will make a few comments which are not directly needed in the coming developments.

a. For non-parallelizable manifolds we will rapidly sketch the modifications to be made on the above argument.

1. First we notice that if we have a finite number of linear maps $s_1, s_2$ of $x$ such that $1 - s_1 - s_2$; then by putting $u = s_1 s_2 x$ and $v = s_2 s_1 x$ we get $1 = s_1 s_2 + s_2 s_1$. Hence it is enough to show that $s_1 s_2$ is something.

b. Now, given a manifold $M$ and any $x \in M$ we can always find a continuous map $F: M \to \mathcal{C}(\mathcal{T})$, $F(x) = 0$ such that for all $y \in V$, a neighborhood of $x$, $F(y) = 0$ and $F_y = s_y$ is an isomorphism. From there we can find a map $F(y) = \text{Diff}(\mathcal{T})$ which is locally transitive near $x$. We will take $F(y)$ to be zero in $\mathcal{V}$ so $F(y) = y$ outside of $\mathcal{V}$. Now using this map $F$ the proof of prop. 10 shall show that $\pi$ is a smooth map in some neighborhood of $x$, i.e., any $\mathcal{C}^\infty$ form with support inside this neighborhood is mapped to a $\mathcal{C}^\infty$ form by $\pi$.

c. We now want to construct a finite number of maps $s_1, s_2$ of $x$ such that we have (1) $1 - s_1 = d_1 + d_2 (1) s_2$ smooth forms inside the open set $\mathcal{U}$ (where $\mathcal{U}$ = M/ (1) and any form with support in $\mathcal{U}$ is mapped into a form with support in $\mathcal{U}$ (same $\mathcal{U}$)). The last can be achieved by reducing the support of the $s_2(x)$ which are used to construct $s_2$. But from (1), (2), (3) it follows at once that $s = s_1 s_2 s_3 \ldots$ will make any $\mathcal{C}^\infty$ form into a smooth form. Thus $1 - s = d_1 + d_2$ is the required parameter.

d. One notes that this patching argument will run into trouble if (with $\mathcal{V}$ Rebecca) we impose the requirement that the image of $\pi$ lies in $\mathcal{C}(\mathcal{T})$. However for filtered manifolds this problem can be avoided. We take a product neighborhood $\mathcal{V}$ and a continuous map $F: M \to \mathcal{C}(\mathcal{T})$ such that $F_y = s_y$ is an
isomorphisms for \( \mathfrak{g} \neq \mathfrak{h} \), \( \mathfrak{g}(\mathfrak{h}) = 0 \), \( \mathfrak{g}(\mathfrak{g}) = \mathfrak{g} \) for \( \mathfrak{h} \neq \mathfrak{g} \) and \( \mathfrak{g} = \mathfrak{h} \), and for all \( \mathfrak{a} \neq \mathfrak{g} \neq \mathfrak{h} \) (\( \mathfrak{a} \neq \mathfrak{g} \) being the filtration). Now \( \mathfrak{g}(\mathfrak{h}) \) will map \( \mathfrak{h} - \mathfrak{g} \) into \( \mathfrak{h} - \mathfrak{g} \) and we will have no trouble in setting up (2) besides (1) and (2). Thus forming a finite number of \( \mathfrak{g}_i \) constructed from each \( \mathfrak{g}_i \) will give the required \( \mathfrak{z} \)-parametric. (Note that if each \( \mathfrak{g}_i \) distorts filtration by 1 unit, so does \( \mathfrak{g}_1 = \mathfrak{g}_2 = \mathfrak{g}_3 = \mathfrak{g}_4 = \cdots = \mathfrak{g}_n \) since each \( \mathfrak{g}_i \) commutes with the filtration.) We now state as

**Proposition 21.** A filtered manifold possesses a \( \mathfrak{z} \)-parametric for \( \mathfrak{g} \).

E. We remark that the hypotheses used in step 1 of the construction hold also if the foliation has differentiable leaves, is regular (i.e., each leaf has a transverse skeleton meeting a leaf only once), and arises from the action of a Lie group which acts freely on the manifold. For such a foliation we can construct maps \( \mathfrak{q} \) with the same properties as in 1. So such foliations setting from the vector action of a Lie group also have a \( \mathfrak{z} \)-parametric.

b. Transverse invariant forms form the complex

\[ \mathfrak{p}, \mathfrak{q}, \mathfrak{r}, \mathfrak{s}, \mathfrak{t}, \mathfrak{u} \]

Harish-Chandra [29] constructed a parameter for \( \mathfrak{p} \) under the condition that one has a bundle-like metric. And then (since \( \mathfrak{p} \) is a metric) he follows that the homology, i.e., \( \mathfrak{p}^{\mathfrak{d}} \), is finite dimensional, under this condition. Since scheme [29] has shown that \( \mathfrak{p}^{\mathfrak{d}} \) must be finite dimensional (31) any \( \mathfrak{d} \) will have a parameter. In any case if we have some \( \mathfrak{d} \) such that \( \mathfrak{d} = \mathfrak{p}^{\mathfrak{d}} \), then they induce maps \( \mathfrak{d}^{\mathfrak{d}^{\mathfrak{d}}} \). In the complex (20) of step 1. These induced maps will obey the identity \( \mathfrak{d}^{\mathfrak{d}} = \mathfrak{d}^{\mathfrak{d}^{\mathfrak{d}}} \). So if the spaces of (20) can be given a Hausdorff topology, then we will have a parameter for this complex also.

c. How we point out the single connection between Hausdorffness and derial's quality. We follow the argument given by Serr [29]. The basic lemma is that if \( \mathfrak{d} \) is a Frechet space and the differential \( \mathfrak{d}^{\mathfrak{d}} = \mathfrak{d} \) is a topological homeomorphism then the homology \( \mathfrak{d}^{\mathfrak{d}}(\mathfrak{d}) \) is also a Frechet space (in the induced topology) and its dual is \( \mathfrak{d}^{\mathfrak{d}}(\mathfrak{d}) \) where \( \mathfrak{d}^{\mathfrak{d}} = \mathfrak{d} \) is the topological dual. This is Lemma 1 in [29]. Let us now take up the case with distributions coefficients as in [29]. This complex (1,6) carries a natural filtration and thus gives another spectral sequence \( \mathfrak{d}^{\mathfrak{d}} \) also converging to the dual mass calmscapely. One may note that \( \mathfrak{d}^{\mathfrak{d}} \) has the
topological dual \( E^{*} \) (some proof by prop. 4 of [23]), and that the dual map \( d_{D} \) corresponds to \( d_{D} \).
\( E_{D} \) under this isomorphism (some proof as 
Prop. 3 of [23]). Using these remarks one gets the 
following result by using Lemma 1 again and again.

Proposition 12. If the spectral sequence 
merging \( E_{1} \),..., are topological isomorphisms (with 
respect to the induced topologies) then Seifert's 
duality \( E_{1}^{*} \cong E_{1}^{*} \) holds (it is of course 
orientable).

This proposition implies that if \( E_{1} \) is finite 
dimensional then Seifert's duality holds, and that all 
compact dimensional 
riemannian manifolds with \( \dim E_{1} = \infty \) are 
orientable manifolds.

d. Calculations made by Kusner and given by 
Reznikov [24] show that for almost all irrational flows 
on the torus \( \mathbb{T}^{2} \), \( \omega \in \mathbb{R} \) (in fact \( \mathbb{T}^{2} \) has \( \mathbb{Q} \) as 
measure \( \mu \) and \( \mathbb{Q} \) as \( \mathbb{Q} \) as \( \mathbb{Q} \) is irrational), 
but not in the rational case where \( \mu \) is irrational 
on the torus for which \( \mu \) is \( \mathbb{Q} \) as \( \mathbb{Q} \) as \( \mathbb{Q} \) is irrational.

So we will recall their arguments below.

For the present however that an irrational toral 
flow is generic (i.e., any measurable set made up of 
complete leaves is of measure 0 or 1) and \( \mathcal{O} \), 
i.e., every leaf is dense). So neither of these 
hypotheses suffice to ensure \( \dim E_{1} = \infty \).

Think of the \( z \) torus \( z^{2} \) as the \((x,y)\) plane 
satisfying \( 0 \leq z \leq 1 \), \( 0 \leq x \leq 1 \) with proper boundary 
identifications. Let us be given a 1-fibration 
represented by straight lines meeting an angle \( \alpha \) with 
the \( x \)-axis, such that \( \tan \alpha = \lambda \) is an irrational number.

Let us denote by \( x \) the direction along the leaves. Let 
us not try to calculate \( \lambda^{1/2} \). Obviously \( \lambda^{1/2} \) can be 
supposed to be all smooth forms \( \psi = \psi_{x} \) with \( \psi \) 
locally \( \lambda^{1/2} \) then \( \psi = \psi_{y} \) where \( \psi \) is another smooth 
function. Thinking of both \( \psi, \chi \) as \( \mathbb{R} \)-periodic functions 
in \((x,y)\) we have their Fourier 
series 
\( \psi = \sum_{n=1}^{N} \exp \left( \pi \mathbf{j} \cdot \mathbf{k} \right) + \sum_{m=1}^{M} \exp \left( \pi \mathbf{m} \cdot \mathbf{k} \right) \), 
with \( \mathbf{j} = \frac{\psi}{2\pi} \) we see that we must 
have \( \mu_{0} = 0 \) and then \( \sum_{0} = 0 \) for \( \psi = \psi_{x} \) 
for \((x,y) \neq (x,0)\). The denominator \( \phi \neq \phi \) as \( \alpha \) is irrational.

Similarly whenever \( \psi_{0} = \psi \) one can 
build a fourier series for \( \psi \). However this series 
does not converge for some values of \( \lambda \). But if \( \lambda \) is 
irrational \( \mu_{0} = 0 \) for \( \psi = \psi_{x} \).

Thus if \( (x,y) \neq (x,0) \) then \( \psi = \psi_{x} \) does converge to a smooth 
function. By a theorem in approximation theory (due to Helson) this condition is satisfied 
for all \( \lambda \) outside a set of measure zero.
13. In this section we shall cover the general relations between connection theory and our spectral sequence.

There are two main (kinds of) definitions of connection. The first may be called "algebraic" as it is convenient in differential geometry. Here a connection on a vector bundle \( \mathcal{V} \) (on \( \mathbb{R}^n \)) is a morphism \( C^\infty(\mathbb{R}^n, \mathcal{V}) \to C^\infty(\mathbb{T}^n) \) obeying certain rules. [See, e.g., Kobayashi and Nomizu.]

Proof. Dröwov has suggested that this example may be extended to nilflows on \( \text{Nil}(n) \) (see Auslander [52], Auslander Studies n° 99).

If the foliation has a dense leaf note that \( \text{dim} \mathcal{L} = n = \text{dim} \mathcal{V} \). This follows since \( \mathcal{L} \to \mathbb{R}^n \) consists of transverse invariant forms, and the value of such form over a leaf is determined by its value at one point thereof.

2. The following question are interesting:

1. Is \( \text{dim} \mathcal{R}_1 = \text{dim} \mathcal{L} \) for all Nilflows? (See a book on dynamical systems for definitions.)

2. Is \( \text{dim} \mathcal{R}_2 = \text{dim} \mathcal{L} \) for all compact foliated manifolds?

As yet, the author is unable to answer these.

14. Let \( \mathcal{V} \) be a smooth vector bundle on \( \mathbb{R}^n \) and denote by \( \alpha(\mathcal{V}) = \mathcal{F}(\mathcal{V}) \) the vector space of smooth sections of \( \mathcal{V} \otimes \mathcal{V}^* \). Let us try (in analogy with the exterior derivative \( d \) ) to build an endomorphism

\[
\alpha(\mathcal{V} \otimes \mathcal{V}) = df \wedge \sigma + \text{flat} \mathcal{V} \otimes \mathcal{V} + \mathcal{F}(\mathcal{V}) \otimes \mathcal{F}(\mathcal{V})
\]

\[
= df \wedge \sigma + \mathcal{F}(\mathcal{V}) \otimes \mathcal{F}(\mathcal{V}) \tag{33}
\]

Here \( f \in C^\infty(\mathbb{R}^n) \), \( \sigma \in C^\infty(\mathcal{V}) \), \( \mathcal{F}(-) \) is \( \mathcal{F}(\mathcal{V}) \), and the meaning of the various products is the natural one. Note that by (33) \( \alpha(\mathcal{V}) \) cannot be the zero map.

Proposition 4. One can find an isomorphism \( \alpha(\mathcal{V}) \cong \mathcal{F}(\mathcal{V}) \) satisfying (33) (i.e., called a connection on \( \mathcal{V} \)).
various products. Hence, by \((31)\), it suffices locally to just define $\Phi^{(\gamma)}(\gamma') = \Phi^1(\gamma) + \Phi^2(\gamma)$, i.e., \(\tilde{\gamma}(\gamma') = \tilde{\gamma}(\gamma) + \Phi^2(\gamma)\). To do this we select, for a basis $\gamma = a_1 e_1, \ldots, a_n e_n$ of $\sigma(\gamma)$, a matrix $\Phi' = \Phi^1(\gamma)$ of 1-forms and put
\[
\Omega = \Phi^2(\gamma)
\]
(33)
If in an overlapping locality a basis $\gamma'$ is chosen with $\gamma' = \gamma_e$, we put $\gamma' = (\Phi, \Phi')$.\(^{16}\)
(34)
Now we check that $\tilde{\gamma} = \gamma + \Phi$, which is $\tilde{\gamma}(\gamma)$. Hence the definition extends globally. Actually it is not compact a finite number of such matrices $\tilde{\gamma}$ (called connection matrices) are enough to define $\Phi$. \(\Box\)

Now we naturally ask if one can ask for $\tilde{\gamma} = 0$.

**Proposition 36.** One can find an endomorphism $\Phi^{(\gamma)}(\gamma) = \tilde{\gamma}(\gamma)$ satisfying \((33)\) if and only if the structure group of $\gamma$ can be reduced to a finite subset.

**Proof:** Let us first take $\tilde{\gamma} = 0$. Choosing as above a basis $\gamma$ for $C^\infty(\gamma)$ in a local area this makes $\tilde{\gamma} = 0$, i.e., $\tilde{\gamma}(\gamma) = 0$ (with above terminology), and hence by \((33)\), $\gamma' = \gamma = u, v, \ldots, w = u$. Hence we get the equivalent condition
\[
\Phi = \Phi^1(\gamma), \quad \Phi^2(\gamma) = 0
\]
(35)
The matrix $\Phi$ of 2-forms (depending on $\gamma$) is called a connection matrix. One can see that in a new basis
\[
\gamma' = \Phi^1(\gamma)
\]
and hence $\tilde{\gamma} = 0$ is a condition quite independent of the local basis selected. By the Ambrose-Singer theorem (see Ambrose and Hurwitz, Chapter II, eqn. p. 52) this implies that the local (or the restricted) homotopy group of $\gamma$ is zero. Locally trivialising $\gamma$ by horizontal sections we can arrange that the coordinate transformations are abelian $\sigma(\gamma)$ matrices. Conversely let us suppose that we can cover $\gamma$ by a finite number of trivialisations $\gamma$ (basis of $C^\infty(\gamma)$) such that the connecting matrices $\Phi$ are constant. Now we take $\gamma'$ as the zero matrix in each of these. Since $\tilde{\gamma} = 0$ the required transformation law holds and we have a connection. It is clear that we have $\tilde{\gamma} = 0$. \(\Box\)

Before proceeding with proving another prop., of the same kind we will indicate some additional results. From a connection $\Phi$ on $\gamma$ we can attempt to compute the cohomology $H^\ast(\gamma, \tilde{\gamma})$, where $\tilde{\gamma}$ denotes the space of gauge over horizontal sections (i.e., $\tilde{\gamma} = 0$) of $\gamma$. In the particular case given by the above proposition this sheaf has a fine resolution by
\[
0 \rightarrow \gamma \rightarrow \Phi(\gamma) \rightarrow \cdots \rightarrow \Phi(\gamma) = 0
\]
and we have a generalised descent theorem. This
cohomology $H^*(x_0, x_1)$ is traditionally called 'with
local coefficients $x_i$.' In the general case when a
resolution is not readily available and one has to
resort to methods of a different kind (see work of
R.

Kakui, Everset, et al.). It is clear that the
theory of characteristic classes is intimately related
to this cohomology.

Now we consider the basic concept of such a resolution
and denote by $P(i, j)$ the vector space formed by the smooth
sections of $F^{*}.F^{*}.0$. We denote by $\phi_{ij}$ the part of
the cohomology which is of bidegree $(i, j)$.

Theorem 50. One can find an endomorphism
$\phi_{ij}(\cdot) = \phi_{ij}(\cdot)$ satisfying (33) and $\phi_{ij}(\cdot) = 0$ if and only
if $\phi$ is an invariant bundle.

Proof. Suppose that $\phi_{ij}(\cdot) = 0$. Hence, with respect to
a local basis $e$ for $C^{*}(\cdot)$, the curvature matrix $\gamma$
consists of $2$ forms having filtration $\gamma_1$. Using the
Atiyah-Bott theorem we see that the local holonomy
of each leaf is trivial. But if we cover $\mathcal{X}$ by a
finite number of trivializations $\mathcal{X}$ open each of
the horizontal slices along leaves it follows that the
connecting matrices are constant along leaves.

Conversely take some trivializations $\mathcal{X}$ which are
related by matrices $\mathcal{X}$ constant along leaves. Let $\gamma$
be the matrices of any connection. Now compare the
parts of (33) of bidegree $(1, 0)$. We get
$\gamma_{ij} = \lambda_0 \phi_{ij}(\cdot)$. These matrices $\gamma_{ij}$ give required
connection.

An invariant bundle shall always, if not
otherwise mentioned, carry such a connection. We shall
call this a $\theta$-connection. The above proof suggests
that for each $\gamma_{ij} \in \mathcal{X}(\mathcal{X})$, $\gamma_0 \in \mathcal{X}(\cdot)$, we shall define
the notion of a $\theta$-connection on $\mathcal{X}$ as follows. Let
us take any couple $(\mathcal{X}, \gamma)$ representing . That is $\mathcal{X}$ is
a covering of $\mathcal{X}$ by open sets $\mathcal{X}_1$ and $\gamma$ consists of
sections $\gamma_{ij}$ over $\mathcal{X}_1$ of the sheaf $\mathcal{X}$ which obey
$\gamma_{ij}(\mathcal{X}) = 0$ (see Matsushita's book for more details).
Now on each $\mathcal{X}_1$ choose trivializations $\mathcal{X}_1$ for $C^{*}(\cdot)$ so
that $\gamma_{ij} = 0$. Let $\mathcal{X}_1$ be the connection matrices with
respect to $\mathcal{X}_1$. If they are of filtration $\gamma_1$ we say
that we have a $\theta$-connection. This is a valid
definition for if some other trivializations $\gamma_{ij}$ are
taken equation (33) shows $\gamma_{ij}$ is also of filtration $\gamma_1$.
Again if some other couple $(\mathcal{X}, \gamma)$ is chosen we have
$\gamma_{ij} = 0$. By (33) we see that $\gamma_{ij} = 0$ where $\gamma_{ij} = 0$.
So we see that $\gamma_{ij} = 0$ where $\gamma_{ij} = 0$. By (33) the
connection matrices with respect to $\gamma_{ij}$ will be also of
filtration $\gamma_i$. The proof above now tells us that
If the invariant bundle $\mathcal{H}$ is associated to $\mathfrak{h}(\mathbb{C}, \mathfrak{g})$, then we have a $\mathfrak{g}$-equivariant connection on $\mathcal{V}$.

We shall denote by $\mathcal{L}(\mathcal{V})$ the holonomy of $\mathcal{V}$ under $\mathcal{H}_{11}$, if $\mathcal{V}$ is a Bott connection. Also denote by $\mathcal{L}(\mathcal{V})$ the $d$-dimensional vector space of all endomorphisms of $\mathfrak{L}^n$. The group $\mathcal{L}(\mathcal{V})$ acts on it by

$$e^{a} = g_{i}^{a}e^{i}.$$  

(30)

Using this action we construct a vector bundle $\mathcal{V}$ with fiber dimension $\mathfrak{m}$ associated to $\mathcal{V}$. It is clear that one can think of the skew-symmetrization (of a connection $\mathcal{L}(\mathcal{V}) = \mathcal{L}(\mathcal{V})$) as an element of $\mathfrak{L}(\mathfrak{m})$ given locally by the curvature matrices; encountered earlier. We denote this global form also by $\mathcal{G}$. So $\mathfrak{L}(\mathfrak{m}) = \mathfrak{L}(\mathcal{V})$.

The $\mathfrak{m}$-invariant bundle $\mathcal{V}$ is a $\mathfrak{g}$-equivariant connection if it is associated to the skew-symmetrization $\mathcal{L}(\mathcal{V})$ of $\mathfrak{g}$, consist of transverse invariant forms. Note that the curvature of any such connection is of the form $\mathfrak{g} \oplus \mathfrak{m}$. In other words $\mathfrak{g}_{11} = 0$, any connection obeying this condition will be called a $\mathfrak{g}$-equivariant connection. If the vector bundle $\mathcal{V}$ admits a $\mathfrak{g}$-invariant connection we shall say that it is $\mathfrak{g}$-stable.

Proposition 31. A bundle $\mathcal{V}$ is stable if and only if $[\mathfrak{g}_{11}] \in \mathfrak{H}^{1} (\mathfrak{m})$ vanishes.

Proof. The "only if" part is obvious. So only the converse needs a demonstration. If $\mathcal{V}$ is a Bott connection and we choose a Bott connection and then look at the holonomy class $\mathfrak{g}_{11}$ lying in $\mathfrak{H}^{1} (\mathfrak{m})$, we assume that it vanishes. Then there exists a one-form $\mathfrak{g} \in \Omega^{1}(\mathfrak{m})$ such that $\mathfrak{g}_{11} + \mathfrak{g}_{11}$.

Choosing local bases $\mathfrak{g}$ which are coordinates over leaves one can think of $\mathfrak{g}$ as being locally defined by matrices varying by $\mathfrak{g} = \mathfrak{g}_{11}$. As if $\mathfrak{g}$ denotes the connection matrix of the given Bott connection, $\mathfrak{g}_{11} = \mathfrak{g}_{11}$, reads $\mathfrak{g}_{11} (\mathfrak{g}_{11}) = \mathfrak{g}_{11}$, i.e., that these matrices consist of transverse and invariant $1$-forms.

This shows that $\mathcal{V}$ is $\mathfrak{g}$-equivariant.

We refer the reader to Deligne [10] for a.
similar viewpoint of connection theory. [He also considers connections as derivations of \(a(v)\) lying above \(a\).] Also the work of Helms [21] is closely related to the above, e.g., he has the notion of an invariant connection, though the reduction of the structure induced is not stressed.

159. This section will be devoted to differentiating the main theorems of section 181 into statements about the Weil homomorphism. Also some essentially new features will be pointed out.

It is well-known, e.g., [17], pages 65-66, that the definition of connection in section 181 (by means of sheafings [23]) is equivalent to putting on \(P\) (the principal bundle of \(H\)) a smooth plane field transverse to the fibres, and of dimension \(m\), which is preserved by the group action. With this in mind we now go over to the 'algebraic' treatment of connections.

Let \(G\) be any Lie group. We first of all introduce the notion of a \(G\)-algebra, say that we mean a graded anticommutative algebra over \(G\) (or more generally, over any commutative ring with unity) which is supplied with a differential \(d\) [i.e., a skew derivation of degree \(-1\) and order \(2\) (\(e^2 = 0\))] and for each \(x, y\) the Lie algebra of \(G\), is supplied with the endomorphism \(l_x\) [which is a skew derivation of degree \(-1\) and order two] and \(l_y^1\) [a derivation of degree zero] such that the following commutative rules hold:

\[
[l_x, l_y^1] = [l_y, l_x^1], \quad l_x^1 l_y^1 = [l_x, l_y^1]
\]

(29)

Note that these imply that \([l_y^1, l_x^1] = 0\).

Now we give an example of a real \(G\)-algebra. Let us take the algebra \(W(1) = \mathbb{R}[x^0, x^1, x^2]\) formed by connecting the exterior and the symmetric algebra generated by \(G\). If we agree to give the grading \(2p\) to polynomials of degree \(p\) it is anticommutative. We will use the notation \(A(0)\) (resp. \(A(0)^0\)) for \(A^0\) (resp. \(A^0(0)\)) in the following. Now we define the \(G\) endomorphisms on the generating elements \(x^0\) and \(x^1\); this will define them everywhere as they are derivations of skew derivations. Note that any (new) derivation will be zero on \(x^0\) and \(x^1\). We now define \(l_x\) for \(x \not\in A(0)^0\),

\[
l_x^a = x^a \quad \text{and} \quad l_x^1 = 0
\]

for \(x \in A(0)^0\); for \(x \in A(0)^0\),

\[
l_x^a = x^a \quad \text{and} \quad l_x^1 = x\partial x^a
\]

for \(x \in A(0)^0\). The endomorphism \(l_x\) with \(l_x^0(1) = x\partial x^a\) and
Let $\varphi: \mathfrak{g} \to \mathfrak{h}$ be the canonical isomorphism $\mathfrak{g}^1(\mathbb{C}) \to \mathfrak{h}^1(\mathbb{C})$. Then for $\eta \in \mathfrak{g}^1(\mathbb{C})$, $(\eta \cdot \lambda) \in \mathfrak{h}^1(\mathbb{C})$ is defined by

$$(\eta \cdot \lambda)((x, t)) = \varphi(\eta)((x, t)) + \varphi(\lambda)((x, t))$$

for $\eta \in \mathfrak{g}^1(\mathbb{C})$.

Note that the last part of (42) means that if $\mathbb{C} \cdot \lambda \in \mathfrak{g}^1(\mathbb{C})$, we refer the reader to Chapter $\mathbb{G}$ for more details regarding these definitions. Since it is also shown that the commutative rules (39) hold, this $\mathbb{G}(\mathbb{C})$ is called the **Lie Algebra** of $\mathbb{G}$. (Note that it is really $\mathbb{G}$ that is important; we can start off with any Lie algebra and do the above construction.)

Now we will define a *connection* to be a $\mathfrak{g}$-algebra morphism from $\mathfrak{h}(\mathbb{C})$ to some other real $\mathfrak{g}$-algebra. Hereby a $\mathfrak{g}$-algebra morphism we mean that the entire $\mathfrak{g}$-algebra structure is preserved under the map.

To point out its relationship to the definition above we first of all see that the space of all smooth vector fields on a principal bundle $\mathbb{P}$ with group $\mathbb{G}$ in fact a $\mathfrak{g}$-algebra. Denote this space by $\mathfrak{A}(\mathbb{P})$. Then $\mathfrak{A}(\mathbb{P}) = \mathfrak{A}(\mathbb{P})$ is the exterior derivative. On the other hand for each $\xi \in \mathfrak{g}$ we get a canonical vector field along the fibres of $\mathbb{P}$. By taking the interior product with respect to this vector field (which shall also be denoted by $\iota$) we define $\iota_{\xi}(\mathbb{P}) = \mathfrak{A}(\mathbb{P})$. Finally, the differentiation with respect to this vector field yields the third commutator. All the equations of [35] are valid by standard results.

Now take any $\mathfrak{g}$-algebra morphism $H(\mathbb{C}) \to \mathfrak{A}(\mathbb{P})$. Being an algebra morphism is determined uniquely by its values on $\mathfrak{g}^1(\mathbb{C})$ and $\mathfrak{h}^1(\mathbb{C})$. But by (42) for any $\eta \in \mathfrak{g}^1(\mathbb{C})$ we have $\varphi(\eta) = \mathfrak{A}(\mathbb{P}) = \mathfrak{A}(\mathbb{P}) - \mathfrak{h}(\mathbb{C})$. Hence it is determined uniquely by its restriction $\mathfrak{g}^1(\mathbb{C}) \to \mathfrak{A}(\mathbb{P})$. This restriction commutes with $\lambda$ and with the maps $\iota_{\xi}(\mathbb{P}) = \xi$. Thus if $\varphi(\mathfrak{g}^1(\mathbb{C}))$ is a basis of $\mathfrak{g}^1(\mathbb{C})$ then $\iota_{\xi}(\mathbb{P}) = \xi \cdot \lambda$ will give us smooth 1-forms on $\mathbb{P}$ which are equivariant under the right action of $G$. If $\mathfrak{A}(\mathbb{P})$ is the exterior derivative of a $\mathfrak{g}$-algebra $\mathfrak{G}(\mathbb{C})$ then for any $\mathfrak{g}$-algebra $\mathfrak{G}(\mathbb{C})$, the required $\mathfrak{g}$-algebra $\mathfrak{A}(\mathbb{P})$ will be the required $n$-dimensional smooth plane field which is transverse to the fibres and which is preserved by the group action. Thus we have rephrased the standard definition of connection recalled above. One can of course restate the argument back and interpret any such plane field as a $\mathfrak{g}$-algebra morphism.
The restriction map \( \beta^i(\xi) \to \xi^i(\eta) \) can be interpreted also as a \( \xi \) valued 2 form on \( \eta \) which is equivalent under right action of \( \xi \). It is the \textit{equivariant} or the connection \( \eta \). When we are thinking of \( \eta \) as foliated by the filtration we will write \( \eta_z \) to distinguish it from the normal case when \( \eta \) is foliated in cosets. We shall use the given filtration on \( \eta \).

**Proposition 52.** The image of \( \xi^i(\eta) \to \xi^i(\zeta) \) lies in \( \xi^i(\eta) \).

**Proof:** The proposition simply states that the curvature form is horizontal.

We shall say that a connection \( \eta(\xi) \to \eta(\zeta) \) is a \textit{flat connection} if the image of its curvature map \( \beta^i(\xi) \to \xi^i(\eta) \) lies in \( \xi^i(\eta) \), i.e., if the curvature is of filtration \( \xi \) with respect to the coset action of \( \xi \).

We shall also say that the connection \( \eta(\xi) \to \eta(\zeta) \) is \textit{flat} if the image of its curvature map \( \beta^i(\xi) \to \xi^i(\eta) \) lies in \( \xi^i(\eta) \), i.e., if the curvature is of filtration \( \xi \) with respect to the coset action of \( \xi \).

We now examine in more detail the relationship between the structure of the invariant bundle \( \eta \) and flat connections.

The isomorphism classes of principal \( \xi \)-bundle over \( \xi \) form the set \( \xi^1(\eta) \) - \text{tors}. Every \( \xi \)-torsion \( \xi \)-structure \( \eta(\xi) \to \eta(\zeta) \) is \textit{flat}. We shall also say that a connection \( \eta(\xi) \to \eta(\zeta) \) is \textit{flat} if the image of its curvature map \( \beta^i(\xi) \to \xi^i(\eta) \) lies in \( \xi^i(\eta) \), i.e., if the curvature is of filtration \( \xi \) with respect to the coset action of \( \xi \).

**Proposition 53.** All \( \xi \)-flat connections \( \eta(\xi) \to \eta(\zeta) \) are in the same 1-chain homotopy class.
Proof. Let $f_1, f_2$ be two such flat connections.

Choose a cocycle $(U_1, U_2, U_1 U_2^{-1})$ of $\mathfrak{g}$ in which both $f_1$ and $f_2$ can be represented by connection forms of filtrations $\mathfrak{g}_+$. We now use the $1$-separation of $\text{Lie}(G)$ (see section 9). Let us follow $\text{Lie}(G)$ in codimension $0$ in the obvious way. We will have the principal bundle $P \times E$ sitting above $\text{Lie}(G)$ in the natural way. We can define a third connection, $P(\text{Lie}(G)) = (1)(P \times E)$ as follows:

$P(\text{Lie}(G)) = f_1 + f_2 - f_1 f_2$.

Now $\mathfrak{g}$ acts on a tangent vector $v$ in $P_1$ by $\mathfrak{g}_+$ and $v$ is the standard vector in $\mathfrak{g}_+$. Further $\mathfrak{g}_+^1(\mathfrak{g})$ and $f_1 + f_2 - f_1 f_2$ is tautological.

Since it obviously commutes with $v$, then for all $\varphi \in \mathfrak{g}_+$, this definition $P(\text{Lie}(G)) = f_1 + f_2$ extends to a $G$-algebra morphism. One sees that with respect to the cocycle above the connection matrices of $P$ will also be of filtration $\mathfrak{g}_+$. So it is in fact a flat connection. We now define the chain homotopy $\text{ad} = \text{ad} f_2 = \text{ad} f_1$ by the formula

$\text{ad} = f_2^T f_1 - f_1^T f_2$ for all $\varphi \in \mathfrak{g}_+$.

Just as in section 10 we compute $\text{ad} = \text{ad} f_2 f_1 - f_1^T f_2$ since $\text{ad} f_2 = f_2^T f_2$ (the $C^2$-commutes with induced maps) and $P(\text{ad}) = d f_2(\text{ad})$ ($\text{ad}$ $P$-commutes with $f_2$).

Hence the right hand side equals $f_2^T f_2 f_2$ $P(\text{ad})$.

For the use of $f_2^T(\mathfrak{g})$, it is clear from the definitions of $P(\text{ad})$ that it equals $f_2^T f_2 f_2$. Again for $f_2^T(\mathfrak{g})$ the curvature map $f_2^T f_2 f_2$ is given by

$P(\text{ad}) = f_2^T f_2 f_2 = f_2^T f_2 f_2 = f_2^T f_2 f_2$.

where $\text{ad}$ denotes horizontalization ($\mathfrak{g}_+$).

P. 79-87), (in the previous $P \times E$ and think of $f_2$, $f_2$ as forms on $P \times E$).

If this is $f_2$, $f_2$ the forms containing $\text{ad}$. Hence $\text{ad} f_2 = f_2^T f_2 f_2$. Thus the relation

$\text{ad} = \text{ad} f_2 f_2$ holds. Finally from the definition of $f_2$ it is clear that it preserves filtration. As $\text{ad}$ is the required 1-chain homotopy, in the terminology of section 9.

An immediate consequence of this result is that for $f_2$, the induced spectral sequence maps $f_2^T(\mathfrak{g}_+; \mathfrak{g}_+)$ do not depend on the choice of the flat connection $f$. We can record this as

\text{Corollary 36.} Each invariant structure $\mathfrak{g} \in \mathfrak{g}_+^{\mathfrak{g}_+}$ gives us a well-defined map

$f_2(\mathfrak{g}_+^T) = f_2(\mathfrak{g}_+^T)$ for $\mathfrak{g}_+^T$ (92)
The dual of Proposition 3 of section 19

**Proposition 30.** A bundle $\mathcal{L}(\mathfrak{g},\mathfrak{c})$ is still

if and only if the map $\mathcal{F}_{\mathfrak{g},\mathfrak{c}}^1(\mathfrak{g}) \to \mathcal{F}_{\mathfrak{g},\mathfrak{c}}^1(\mathfrak{c})$ vanishes.

**Proof:** This map becomes simply $\mathcal{L}(\mathfrak{g},\mathfrak{c})$ if we interpret

as in the definitions of section 19.

In the present terminology a connection

$$\kappa_t(v) = \kappa_t^i$$

is called **invariant connection** if the

image of its curvature map $\kappa^t_i(\mathfrak{g}) \to \mathcal{F}_{\mathfrak{c}}^1(\mathfrak{c})$ lies in

$\mathcal{F}_{\mathfrak{c}}^1(\mathfrak{c})$. (Note—by Prop. 34—that any connection is

invariant with respect to the filtration of $\mathfrak{g}$ arising

from the filtration.) We now define the **filtration**

of $\mathfrak{c}$ by setting $\lambda(t) = (\lambda_{\mathfrak{g}}(t) \mathfrak{c}) = (\lambda_{\mathfrak{c}}(t) \mathfrak{c})$.

It follows from Proposition 33 that the differential

of $\kappa_t(v)$ also preserves this filtration. The

following proposition is obvious from the preceding

developments.

**Corollary 35.** Each principal $\mathfrak{g}$-bundle $P$ is

stiff if and only if there is a connection $\kappa^t(\mathfrak{g})$

on $\kappa^t(\mathfrak{c})$ connecting with the filtration (i.e., an

invariant connection).

Further a detailed examination analogous to

the above is possible. Take an element $\xi \in \mathcal{H}(\mathfrak{g},\mathfrak{c})$

and let $\kappa$ be an $\xi$-invariant connection. To emphasize

this relationship we use the notation $\kappa(\xi)$.

**Proposition 36.** All $\xi$-invariant connections

$\kappa(\xi): \mathfrak{g} \to \mathfrak{c}$ lie in the same $\xi$-stable homotopy

class.

**Proof:** We proceed exactly as in the proof of Prop. 35

except that we now employ the filtrations of $\mathfrak{g}$ and $\mathfrak{c}$ (see

section 35). With these changes $\mathfrak{g}$ will also be an

invariant connection. Now the claim homotopy $\kappa$

disturbs filtration by one unit.

This result generalizes a well known theorem

of Voss. First, we see that the induced spectral

sequence maps $\pi^r(\mathfrak{g},\mathfrak{c}) \to \pi^r(\mathfrak{c})$ for $r \geq t$, do

not depend on the choice of the invariant connection

$\kappa(\xi)$, we thus have the following.

**Corollary 37.** Each principal $\mathfrak{g}$-structure $\kappa^t(\mathfrak{g},\mathfrak{c})$

gives a well-defined map

$$\Xi^r(\mathfrak{g},\mathfrak{c}) \to \pi^r(\mathfrak{c}),$$

for $r \geq t$. (144)

To get the classical case we assume that we

have the point filtration on $\mathfrak{c}$, One can see easily

from the definition of the Voss algebra that

$$(2\lambda_{\mathfrak{g}}^t(\mathfrak{g}) \mathfrak{c})$$

are simply the symmetric invariant polynomials of $\mathfrak{g}$—see (144).

Also it is clear that

$$\kappa^t(\mathfrak{g}) = \kappa^t(\mathfrak{c}),$$

and so, for a point filtration

$$\kappa^t(\mathfrak{g}) = \kappa^t(\mathfrak{c}).$$

Thus in this case $\Xi^r(\mathfrak{g},\mathfrak{c}) \to \pi^r(\mathfrak{c})$
is simply the Given well formability. Note that for a point relation $y = f(x)$ and so a $c^1$-map of a $c^0$-map is any differentiable structure for $T$ over $R$.

Riemann-Cartier [16] also interpret well morphisms as spectral sequence morphisms, but they do not give the homotopy invariance results (Propositions 35 and 39).

We shall end this section by pointing out that one can employ the Given formality condition in the above discussion. If $x = (x_1, \ldots, x_k)$ is a point connection, we note that $(x'_{(k+1)}) = 0$; thus we can "throw away" terms involving polynomials of degree $\geq 2k$.

More precisely, we replace $W(\cdot)$—whenever we are dealing with both connections—by the quotient $W(\cdot)/\mathcal{P}_k(\cdot)$. The rest of the treatment is precisely as above. The advantage of this is that though $\kappa_{(n)} = 0$ without truncation (see [5]), where it is known that $\kappa_{(0)} = 0$, $\kappa_{(n)} \neq 0$ with this truncation.

In fact one can compute $\kappa_{(n)}$—with modification—to be equal to the Paul: the rank of the power vector fields on $\mathbb{R}^n$ (see, e.g., Theorem 3.1 in Guillemin's paper in Advances in Mathematics, 1972).

In fact condition, too, both commutators have apparently the problem of understanding the "easy" characteristic classes from the viewpoint of this Generalized cohomology.

35. The most important example of an invariant bundle is the bundle $\mathcal{F} \to \mathcal{T}^n$ of vector fields which all have the invariance distribution $\mathcal{D} = \mathcal{V}$. Any connection on a subbundle of $\mathcal{F}$ or $\mathcal{T}^n$ can be extended to the whole bundle; it is then linear connection—i.e., connections defined on the principal bundle are not tangent frames—that shall concern us now. We shall not assume that $\mathcal{D}$ is necessarily involutive. Any connection on $\mathcal{T}^n$ reducible to $\mathcal{D}$, such that the non-part of the curvature is of filtration $\mathcal{D}$. $\mathcal{D}$ shall be called a non-connection. Here $\mathcal{D}$ is the curvature

36. Proposition. Any connection on $\mathcal{T}^n$ which is reducible to $\mathcal{D}$ and which has zone torsion must be a non-connection and then $\mathcal{D}$ is involutive.

Proof. Without recalling the definition of torsion we simply recall one of the consequences of "extra-connection," i.e., the equation

$$\mathcal{A} = 0 \quad (70)$$

[See Cor. 5 in Kostant and Bismut, Ch. III.] Here $A$ denotes alternation and $\mathcal{D}$ denotes covariant differential. Now use this equation with a section
of $\mathbb{D}$ and $X \in C^{(2)} (\mathbb{D})$, $y \in C^{(2)} (\mathbb{D})$.

$$\text{deg}(x, y) = \text{deg}(y) - \text{deg}(x)$$

Since our connection is reducible to $\mathbb{D}$ both $y_0$ and $y_1$ are sections of $\mathbb{D}$. Thus $(y_1, y_2) = 0$. Thus

$$(y_1, y_2) = -\text{deg}(x, y)$$

whenever $x \in C^{(2)} (\mathbb{D})$, $y \in C^{(2)} (\mathbb{D})$. But eqn. (46) already expresses the condition by which Riemann first defines a connection whose curvature has filtration $\xi$. Thus $x$ is involutive provided $\xi$ is involutive. But if we take $y \in C^{(2)} (\mathbb{D})$ in (46), we get $y(x, y) = 0$, i.e., $y_x[y_0, y_1] = 0$ (as $y(x, y) = 0$ for all $x \in C^{(2)} (\mathbb{D})$). An $(x, y) \in \mathbb{D}$ proving the required involutivity.

The proof exhibits clearly the following

**Corollary 9:** If $\xi$ defines a connection reducible to $\mathbb{D}$ and obeying (46) if and only if $\xi$ is involutive.

We shall say that a connection $\phi$ (resp. $\xi$) of zero torsion which is reducible to $\mathbb{D}$ (resp. $\xi$) is a **linear connection**: for the reason for this name, see (17). Then we can extend the above corollary to the following

**Proposition 12:** A linear connection exists if and only if $\xi$ is involutive.

**Proof:** Clearly it suffices to prove that a linear connection exists on $\mathbb{D}$ when $\xi$ is involutive ($\xi$)* are associated to the same principal bundle. The induced connection on $\mathbb{D}$ will supply a linear connection there. Denote by $F$ the fiber-bundle with fiber $\mathbb{D}$, structure group $\text{GL}(\mathbb{D})$, and coordinate transformations

$$\psi = \frac{\partial}{\partial x} \psi \frac{\partial}{\partial x} \psi$$

as we go from local coordinates $x_1$ to $x_2$. A linear connection is a section of $\mathbb{D}$. One has the relation

$$\frac{\partial}{\partial x} \psi \frac{\partial}{\partial x} \psi$$

with the first condition. (See Ehrenpreis and Hörmander, [17], Ch. III, section 7 for more details.) Since we want our connection to be reducible to $\mathbb{D}$ for $1 \leq i < j$ the $\xi$ and should not contain terms with $k > 1$. So

$$\frac{\partial}{\partial x} \psi \frac{\partial}{\partial x} \psi$$

while the condition for zero torsion is $\psi$. (c)$k$.

$$\psi = \frac{\partial}{\partial x} \psi$$

Both these conditions are compatible with (46). Thus we get a subbundle $\mathbb{D}$ of $\mathbb{D}$ whose fiber is $\mathbb{D}^{(1)} (\mathbb{D})$. Choose any section of this space.

This proof is due to Willems [18], using the equations (49) and (50) above we can give another
proof of the fact that a Walker connection must be a Betti connection (Prop. 41). In fact if one puts [17],
\begin{equation}
\frac{\partial^2}{\partial x^i \partial x^j} - \frac{\partial^2}{\partial x^j \partial x^i} + \frac{\partial^2}{\partial x^i \partial u} - \frac{\partial^2}{\partial x^j \partial u} = 0.
\end{equation}
then the curvature matrix is
\begin{equation}
\frac{\partial^2}{\partial x^i \partial x^j} \delta_{\alpha}^\beta = 0 \quad \forall \alpha, \beta.
\end{equation}

In the given coordinates \( x_1, \ldots, x_\alpha \) compatible with the foliation. Value \((19), \alpha_j^i \beta_k = 0 \) for \( i < 1 \).

\( \beta = 1 \), \( \alpha = 1 \), \( \beta = 0 \) for these values. Using (19) and (20) one sees that \( \alpha_j^i \beta_0 = 0 \) for \( i > 1, j > 0, \) and \( \beta = 1 \); \( i > 1 \) the \( \delta_{\text{forms}} \) are of filtration \( \geq 1 \).

We call a connection which satisfies the hypothesis of (19) \( \gamma \) a basic connection. For a linear connection

Walker = basic = Betti.

So far we have looked for cohomological obstructions by using basic connection only. By Proposition 41 it

seems reasonable to see if a Walker connection leads to anything new.

For this purpose we restate (41).

Given a submalle \( \mathcal{D} \subset \mathcal{F} \) let \( \mathcal{F} \subset \mathcal{G}(\mathcal{D}) \) be the principal bundle of frames whose first 1 entries span \( \mathcal{D} \). The group of this bundle is denoted \( \mathcal{G} \) (all automorphisms

\( \mathcal{G} = \mathcal{G} \) keeping \( \mathcal{D} \) invariant). The bands of affine frames \( \mathcal{A}(\mathcal{D}) \) is a \( n \)-dimensional vector bundle over \( L(x) \).

Let \( P \) denote the part sitting over \( P \), and \( \mathcal{G} \) the group of \( \mathcal{G} \). It is clear that \( \mathcal{G}(\mathcal{D}) - \mathcal{G}(\mathcal{G})(\mathcal{D}) \)

with \( \mathcal{G}(\mathcal{D}) \) is the subalgebra of \( \mathcal{G}(\mathcal{G}) \). Let \( \gamma \) be the connection form and denote by \( \mathcal{G}(\mathcal{D}) - \mathcal{G}(\mathcal{G})(\mathcal{D}) \)

the \textit{conclusion form}. i.e., for a tangent vector \( \mathcal{D} \) at \((x, v), \ldots, (x, v)\) of \( \mathcal{D}, \langle \alpha \rangle \neq 0 \). Then \( \langle x \rangle \) is a

basis of \( T_x \mathcal{D} \). The vector \( \langle x \rangle \Delta \mathcal{D} \ldots \Delta \mathcal{D} \) is in a basis of \( T_x \mathcal{D} \). The vector \( \delta \mathcal{D} \ldots \Delta \mathcal{D} \) also lives under \( \mathcal{D} \). So \( \delta \) and \( \delta \) give us a map \( \delta : \delta \mathcal{D} \ldots \Delta \mathcal{D} \) and \( \delta : \delta \mathcal{D} \ldots \Delta \mathcal{D} \).

Thus we get an algebra morphism \( \delta : \delta \mathcal{D} \ldots \Delta \mathcal{D} \). Again

we have the connection \( \delta : \delta \mathcal{D} \ldots \Delta \mathcal{D} \) and the related \textit{well map}.

\textbf{Proposition 41.} \( \mathcal{D} \) is involutive if and only

if the algebra morphism \( \delta : \delta \mathcal{D} \ldots \Delta \mathcal{D} \) be lifted
to the well map \( \delta : \delta \mathcal{D} \ldots \Delta \mathcal{D} \) for some \( \gamma \).

Proof: By prop. 14, Kobayashi and Nomizu, section 7, Ch. III (17) the curvature \( \delta \) of \( \delta \mathcal{D} \ldots \Delta \mathcal{D} \) acts over \( \mathcal{G} \) and \( \delta \mathcal{D} \ldots \Delta \mathcal{D} \) where \( \delta \mathcal{D} \ldots \Delta \mathcal{D} \) is the torsion. So we use prop. 41 to get.

Suppose more generally that \( \delta \mathcal{D} \ldots \Delta \mathcal{D} \) is a torsionless connection which is reducible to a sub-

bundle with fiber \( \delta \mathcal{D} \ldots \Delta \mathcal{D} \). Then we say that we have a \textit{torsionless} \textit{structure on} \( \mathcal{D} \). The
A study of such structures covers most of geometry. (see prop. 4) a relation is in such a structure. The problem of finding necessary conditions for a torsionless $G$-structure to exist is of great interest. (the most varying theories should be seen in this context.)

According to a result of Cartan, Kobayashi et al. [18] if $G$ is one of the groups

$$GL(n), O(n), U(n), SU(n), SL(n), SU(n)$$

we always have such torsionless connections (affine, riemannian, conformal structures; when parametrized times respectively), such structures shall be called

integral since there is no obstruction. Conversely it is known that if all $G$-structures on $M$ can be made torsionless, then, for $n > 2$, $G$ must lie in the above list (55). In fact for the group $O(n)$ of automorphisms of $\bar{G}$ preserving a non-degenerate scalar product we have seen every $O(n)$-structure can be made torsionless by a unique connection and conversely if every $O$-structure on $M$ admits such a unique zero torsion connnection, then $G = O(n)$.

Another famous torsionless structure occurs when $G$ is composed of matrices of the type $\left(\begin{smallmatrix} A & 0 \\ 0 & I \end{smallmatrix}\right)$. This is called a complex structure on $M$. We suppose $n$ is even; this subgroup of $GL(n)$ is called $\mathbb{C}GL(n)$.

Again, if out of these matrices we take those which are orthogonal we get a still smaller group $O(n)$. A torsionless structure with this group is called a complex structure. It is known that then we have a closed $\omega$-form which represents an integral cohomology class if and only if the manifold is algebraic (Kodaira's theorem). These examples thus show the great importance of torsionless $G$-structures.

By an integral $G$-structure we mean that we can cover $M$ by charts so that the Jacobians lie in $G$. The Frobenius theorem thus says that a torsionless $GL(n)$-structure is an integrable $GL(n)$-structure; while the Frobenius-Grauert theorem makes the same statement with the subgroup $GL(n)$. But a torsionless $GL(n)$-structure is of course not integrable; we need the vanishing of another tensor, the curvature tensor.

The general problem of finding necessary and sufficient conditions for the integrability of a $G$-structure (in terms of the vanishing of certain tensors) has been pursued by Spencer, Bott, and others.

In this section we shall show how the existence of certain torsionless $G$-structures enables us to construct certain $G$-structures.
a. As before, let \( H \) be a foliation. So we can assume that we have a torsionless \( \mathfrak{g}(1,\mathbb{D}) \) structure on \( H \). For the Lie algebra \( \mathfrak{g}(1,\mathbb{D}) \) of this group consists of homomorphisms \( \mathfrak{g}^1 \to \mathfrak{g}^2 \) which preserve \( \mathfrak{g}^3 \). (We think of \( \mathfrak{g}^3 \) as \( \mathfrak{g} \otimes \mathbb{R} \) as usual.) So we now define a smaller Lie algebra \( \mathfrak{g}(1,\mathbb{R}) \) consisting of homomorphisms \( \mathfrak{g}^1 \to \mathfrak{g}^2 \) whose image lies in \( \mathfrak{g}^1 \). We assume that we have a torsionless \( \mathfrak{g}(1,\mathbb{D}) \) structure on \( H \). Let \( \mathfrak{g} \) be a Lie subgroup of \( \mathfrak{g}(1,\mathbb{D}) \) whose Lie algebra lies in \( \mathfrak{g}(1,\mathbb{R}) \). (Examples: If there exists a globally defined vector field \( \xi^1, \ldots, \xi^n \) transverse to the foliation \( \eta \) such that we can cover \( H \) by neighborhoods \( \mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_n \) each that \( \sum_{i=1}^n \frac{\partial}{\partial \xi^i} = 0 \).)

Let us be the principal bundle in question (it is a sub-bundle of the bundle \( \mathfrak{g} \) of frames compatible with the foliation) then we shall describe it by pulling back the foliation on \( \mathbb{R}^n \).

Points of \( \mathfrak{g} \) are of the type \([x_0, x_1, \ldots, x_n] \) where \( x_{\xi^i} \) and \( x_{\eta^i} \) are \( \delta \)-frames for \( \mathbb{R}^n \). For each \( \xi^i \in \mathfrak{g} \) we now define a vector field \( \xi^i \) on \( \mathfrak{g} \) in the following way: \( \xi^i \) at \([x_0, x_1, \ldots, x_n]\) is the horizontal vector which lies above \( \eta^i \) in \( \mathfrak{g}(1,\mathbb{D}) \). (In the terminology of Misner and Sezuke, \( \eta^i \) is a canonical horizontal vector field.) Let us suppose that our connection is \( \mathfrak{g}(1,\mathbb{D}) \)-connection, i.e., that the vector field \( \eta^i \) generates a one-parameter group \( \mathfrak{g}(1,\mathbb{D}) \) of diffeomorphisms of \( \mathfrak{g} \).

Proposition 4. For each \( \eta \in \mathfrak{g} \) the diffeomorphism \( \mathfrak{g}(1,\mathbb{D}) \to \mathfrak{g} \) maps leaves into leaves.

Proof. For each \( \xi \in \mathfrak{g}(1,\mathbb{D}) \) we get a canonical vertical vector field \( \xi^i \) in \( \mathfrak{g} \) (see e.g., [17]). Then \( \mathfrak{g}(1,\mathbb{D}) \) is a Lie group and \( \xi^i \) is a basis \( \xi_1, \ldots, \xi_n \) shall be a basis for the fiber space of \( \mathfrak{g}(1,\mathbb{D}) \). By Prop. 2.3, Ch. III of [17] we have the relation \( [\xi^i, \xi^j] = [\xi^i, \xi^j] \); and since parallelism is seen we see from Prop. 3.4, Ch. III of [17] that \( \xi^i \) is always vertical. Since \( \mathfrak{g}(1,\mathbb{D}) \to \mathfrak{g} \) is a fiber map \( \mathfrak{g}(1,\mathbb{D}) \), we have the first of the above equations tells us that \( \mathfrak{g}(1,\mathbb{D}) \) maps \( \xi^i \) into a vector \( \xi^i \) to a leaf \( \mathfrak{g} \). The second says that \( \mathfrak{g}(1,\mathbb{D}) \) maps \( \xi^i \) (for \( \xi \in \mathfrak{g} \)) into a vector \( \xi^i \) to a leaf \( \mathfrak{g} \). Then the proposition follows.

Now on \( \mathfrak{g} \) we have a complete global parallelization (in the sense of p. 68) given by \( \xi^i \) \(\mathfrak{g}(1,\mathbb{D}) \to \mathbb{R}(\mathfrak{g}) \) for \( \eta^i \in \mathfrak{g}(1,\mathbb{D}) \), where

\[
\xi^i = \eta^i \quad \text{for} \quad \eta^i \in \mathfrak{g}(1,\mathbb{D}) \quad \text{and} \quad \mathfrak{g}(1,\mathbb{D}) = \mathfrak{a} \quad \text{for} \quad \mathfrak{g}(1,\mathbb{D})
\]
We have already pointed out in Propositions 21 and 27 the relation between the existence of a 2-parametric line of simultaneity and zero duality. Note that although $G$ is not compact, its automorphism can be calculated using forms having a compact support; and the map $s$ will be compact on the space of such forms. Another remark to be made in that $M^0_1(1) = E_1^0_1(1)$.

Hence if the foliation arises from a torsionless $G$-structure with $\mathfrak{g} = \mathfrak{g}(1,1)$, then $E_1^0_1(1)$ is finite-dimensional.

We now consider the more general case of a torsionless $G(1,1)$ structure (i.e., a foliation), with the bundle $\mathcal{F}$ of frames. Note that $\mathfrak{g}(1,1)$ decomposes as $\mathfrak{g}(1,1) \oplus \mathfrak{g}(1,1)$ where both parts are Lie algebras and the first part is preserved by frames with respect to the second. Choose any basis $a_1, \ldots, a_n$ of $\mathfrak{g}(1,1)$ agreeing with this decomposition. Also choose a basis $a_1, \ldots, a_1$ of $\mathfrak{g}(1,1)$ agreeing with the decomposition $\mathfrak{g}(1,1) = \mathfrak{g}(1,1) \oplus \mathfrak{g}(1,1)$.

We extend $\mathfrak{g}$ to be the 1-in-dimensional plane field spanned by $a_1, \ldots, a_1; a_1, \ldots, a_1$. Then the following proposition gives us a 1-in-dimensional foliation of $\mathcal{F}$ sitting over the foliation of $\mathfrak{g}$.

**Proposition 22**. The plane field $\mathfrak{g}$ is
Inevitable.

Proof: We know that \( [\alpha, \beta] = [\alpha_1, \beta_1] \) and that \( [\alpha', \beta'] = (\alpha')^* \). So it only remains to show that if \( \alpha_1, \beta_1 \) then \( [\alpha_1, \beta_1] \) is a linear combination of the \( \alpha_1 \) and \( \beta_1 \). Since torsion is zero, by (5), (7), (17), this is a vertical vector. Hence it is enough to show that \( [\alpha_1, \beta_1] \) lies in \( \mathfrak{g}(1.3) \). Here \( \mathfrak{g} \) is in the connection form, and where \( \mathfrak{g}(x) = \mathfrak{g}_x \). Note \( \alpha_1, \beta_1 \) being horizontal vectors, we see from (7), (17), (17), that

\[
\mathfrak{g}(\alpha_1, \beta_1) = \mathfrak{g}(\alpha, \beta) \quad (27)
\]

where \( \mathfrak{g} \) is the curvature form. But the \( x \times x \) part of \( \mathfrak{g} \) is of filtration \( m \) by (5). Hence the right side lies in \( \mathfrak{g}(1.3) \).

Now the question arises whether \( \mathfrak{g}(x) = \mathfrak{g} \) preserves this foliation of \( \mathfrak{F} \). In general, if \( \omega \) is a constant vector field on \( \mathfrak{F} \), then it is \( \mathfrak{g} \). Then one can assume that the torsion-free connection is invariant restricted to \( \mathfrak{F} \).

Proposition 1. \( \mathfrak{F}(x) = \mathfrak{F} \) maps leaves into leaves for each \( x \in \mathfrak{F} \), if the connection is invariant. Also, the above statement is true for the differentiable class \( \mathfrak{g}(x) = \mathfrak{g} \).

Proof: In this case the \( x \times x \) part of the curvature is of filtration \( \mathfrak{g} \). As the end of (27) lies in \( \mathfrak{g}(1.3) \), if \( \alpha_1, \beta_1 \) then \( \mathfrak{g}_x \). Hence \( \mathfrak{g}(\alpha_1, \beta_1) \) is a linear combination of \( \mathfrak{g}_x \) and \( \mathfrak{g}_x \). Again, if \( \alpha, \beta \), then \( \mathfrak{g}(\alpha, \beta) \) is a linear combination of \( \mathfrak{g}_x \) and \( \mathfrak{g}_x \). Hence \( \mathfrak{g}(\alpha_1, \beta_1) \) lies in \( \mathfrak{g}(1.3) \). This shows that \( \mathfrak{g}(x) = \mathfrak{g} \). And thus we have the second part.

Note that conditions which can be supplied with a bundle-like metric are a further invariant, but the converse is not true.

Proposition 2. In case the foliation is invariant we can find a smoothing map \( \mathfrak{d} \mathfrak{g} \mathfrak{d} \mathfrak{g} \) which preserves the foliation (t.g. to (5)) together with a parametrization \( \mathfrak{d} \mathfrak{g} \mathfrak{d} \mathfrak{g} \) which distorts the foliation by one unit. These are related as usual by \( \mathfrak{d} \mathfrak{g} \mathfrak{d} \mathfrak{g} \) (5). Proof: Use the formula (5) and (5), with \( \mathfrak{d} \mathfrak{g} \mathfrak{d} \mathfrak{g} \) replaced by \( \mathfrak{d} \mathfrak{g} \mathfrak{d} \mathfrak{g} \), together with the assumption of \( \mathfrak{d} \mathfrak{g} \mathfrak{d} \mathfrak{g} \), (5).

Again this \( \mathfrak{d} \mathfrak{g} \mathfrak{d} \mathfrak{g} \) will have the familiar consequences regarding the flattening of the \( \mathfrak{d} \mathfrak{g} \mathfrak{d} \mathfrak{g} \) terms of the foliated manifold \( \mathfrak{F} \).
Let us note that prop. 43 implies that all the leaves in \( P \) are diffeomorphic to each other. Instead of \( P \) we could work in the bundle \( P' \) of frames \( \mathbb{R}^\mathcal{D} \) (i.e., the principal bundle of \( \mathcal{D} \)), where this foliation would collapse to the horizontal \( \mathcal{D} \)-dimensional foliation: for each \( \mu \in \mathcal{D} \) the canonical vector field \( \mu' \) in \( P' \) will preserve this foliation. This yields the theorem of Hannant [25]. Making [25] a new foliation which can be endowed with a complete\*\footnote{\textit{\textbf{Note:}} The foliation is complete if it is possible to define a complete vector field along each leaf.}\ \textit{natural connection} can be recovered by noting foliation of the zero direction with diffeomorphic leaves.

\( \mathcal{D} \). Let \( P \) be a principal bundle with group \( G \) sitting over \( \mathcal{K} \). Now \( G \) acts freely on \( P \) from the right, and so for each \( g \in G \) we have a diffeomorphism

\[ R_g \]. We denote by \( H_g \) the vector space of smooth forms on \( \mathcal{K} \), and by \( H_g \) the subspace of \textit{right invariant} forms, i.e., maps \( \phi \) such that \( R_g \phi = \phi \) for all \( g \in G \).

Clearly if \( H_{g} = \psi \) then \( H_{g^{-1}} = \psi \), since \( H_g \) is a subspace of \( H_g \).

Now let us equip \( H \) with a \textit{left invariant} normalized linear measure \( \mu_\mathcal{D} \). Then we define a linear map \( J_\mathcal{D} \) to \( H \) by

\[ J_\mathcal{D} = \int_g H_\mathcal{D} \mu_\mathcal{D}(\mathcal{D}) \]  

\[(50)\]

\( \text{(We will assume for the time being that } \mathcal{D} \text{ is compact--which is a severe restriction. Later on these definitions will be amended for more general cases.)} \)

\[ \text{Note that } J_\mathcal{D}(\mu) = \int_g \phi^* J_\mathcal{D}(\phi) \]

\[ \quad - \int_g \phi^* H_{\mathcal{D}}(\phi) - \int_g \phi^* H_{\mathcal{D}}(\phi)(\phi), \text{ on the measure is left-invariant, and we equal } \text{Arv as } \text{Arv is right-invariant as stated above. Also from (50) it is clear that} \]

\[ \text{J}_\mathcal{D}(\mu) - \text{Arv} \]

\[(51)\]

\( \text{It is clear also that } \text{Arv is a continuous map.} \)

\( \text{Suppose given a parametrization for } A_\mathcal{D}, 1:1, 2 \)

\[ \text{maps } A_\mathcal{D} \xrightarrow{\alpha^*} A_\mathcal{D}, \text{ such that } \alpha \text{ is smoothing and} \]

\[ \text{1 - } \phi^* \phi \text{ as } 0. \text{ Then we can define } 2 \text{ maps } A_\mathcal{D} \xrightarrow{\alpha} A_\mathcal{D}, \text{ by} \]

\[ \alpha^* = \alpha^* \phi \phi. \text{ By virtue of (50) it is clear that we will still have } 1 - \phi^* \phi \text{ as } 0, \text{ and also } \phi \text{ will be a smoothing map. In other words by composing with } \text{Arv we can turn a parametrization for } A_\mathcal{D} \text{ into a parametrization} \]

\[ \text{for } \phi \text{.} \]

\( \text{(Since } \phi \text{ is not compact we take a linear measure} \]

\( \text{on } G \text{ and replace (50) by} \]

\[ \text{Arv} = \int_{\phi G} \phi \text{Arv} \phi \mu_{\phi}(\phi) \]  

\[(51')\]

\( \text{where } G \text{ is a finite measure subset of } G \text{ which } 0 \)
as \( s \to \infty \), clearly the limit will be finite if we work only on bounded forms in \( \mathcal{F} \), i.e., we have a map \( \text{av}_{\text{rot}}^s \to \mathcal{L} \). The equation (39) will also hold. By composing with \( \text{av}_{\text{rot}}^s \) we will be able to change any parametrization on \( \text{av}_{\text{rot}}^{s+1} \) to one on \( \mathcal{L} \).

We now return to the case when we were treating above. \( \mathcal{L} \) is foliated. \( \mathcal{F} \) is the principal \( \mathcal{G}(1,0) \)-bundle of compatible tangent frames and is also foliated by the foliation of prop. 67. Let us assume that we can find a parametrization \( \phi(s) \) of \( \mathcal{F} \) such that \( \phi(s) \) preserves the foliation while \( \phi(s) \) destroys it by one unit (we accomplished this when \( \mathcal{L} \) carries a complete invariant connection). Then the following proposition will allow us the same kind of parametrization on \( \mathcal{L} \).

**Proposition 66.** The map \( \text{av}_{\text{rot}}^s \to \mathcal{L} \) preserves the filtration given by the foliation of prop. 67.

**Proof.** It will suffice to check that \( \text{av}_{\text{rot}}^s \to \mathcal{F} \) preserves the plane field \( \mathcal{H} \) for each \( s \in \mathbb{R} \). For \( \forall \xi \in \mathcal{H}, \mathcal{H} \cdot \xi = \{ e^{s} \cdot \xi \} \cdot (\mathbb{C}, \text{lin}(1+i)) \). Assume \( \xi \in \mathcal{H} \) will lie in \( \mathcal{F} \) also. On the other hand if \( \xi \in \mathcal{F} \), then \( \xi \cdot \xi \) will lie in \( \mathcal{F} \) also. Let \( D \) be the connection form on \( \mathcal{F} \); then \( D\xi = \mathcal{H}(\xi) \). Let \( u \) be the connection form on \( \mathcal{F} \). We have \( u(\xi, \xi) = \mathcal{H}(\xi) \cdot \xi \cdot \xi \). (By 1-11-11, 171). The last term lies in \( \mathcal{F} \) as \( \mathcal{H}(\xi) \) is an ideal in \( \mathcal{F} \). Hence the vertical vector \( \mathcal{H}(\xi) \) lies in \( \mathcal{L} \). CED

Let us denote the spectral sequence resulting from \( \mathcal{L} \) by \( E_r(z) \). The existence of a parametrization for \( \mathcal{L} \) gives us invaluable information about \( E_r(z) \), e.g., that \( E_r^2(z) \) is finite dimensional.
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